Renormalization of 2-loop diagrams

In the previous lecture, we set up renormalized perturbation theory for Φ^4 theory. We wrote the set of Feynman rules

 $= \dfrac{i}{p^2 - m^2}$  $= -i\lambda$

 $= i(p^2\delta_Z - \delta_m)$  $= -i\delta_\lambda$

The counterterms $\delta_Z, \delta_m, \delta_\lambda$ are to be determined by conditions such as

$\rightarrow$  $\rightarrow \; = \; -i\,\mathcal{M}(p^2)$

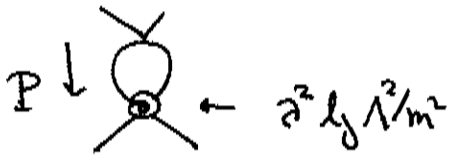
$$\mathcal{M}(p^2)\Big|_{p^2=m^2} = 0 \qquad \frac{d}{dp^2}\mathcal{M}(p^2)\Big|_{p^2=m^2} = 0$$

and  $= 0$ at threshold $s = 4m^2$

We showed explicitly at the 1-loop level that these conditions determine $\delta_Z, \delta_m, \delta_\lambda$ in such a way that all divergences are removed from the theory. I claim that this is actually true to all orders in perturbation theory. However, there is a subtlety. Consider the diagram

In our simple discussion of the degree of divergence, we estimated that this diagram has one constant term that is $\log \Lambda$ divergent and is ~~otherwise~~ otherwise finite. However, that is not correct. Consider what happens when the loop momentum $k \to \infty$

Then the lower loop gives a divergent constant and the diagram can be approximated as



Now integrating over the outer loop gives

$$\sim \lambda^3 \log \Lambda^2/m^2 \int_0^1 dx \, \log\left(\frac{\Lambda^2}{m^2 + x(1-x)p^2}\right)$$

so this diagram has a constant part that diverges like $(\log \Lambda^2)^2$ — which still can be absorbed by $\delta\lambda$ — but also a <u>nonlocal</u> <u>divergence</u>

$$\sim \lambda^2 \cdot \log \Lambda^2 \cdot \log p^2 \quad \text{for } p^2 \gg m^2$$

This <u>cannot</u> be absorbed by  .

But, fortunately, if we do perturbation theory will the counterterm vertices, we find a diagram with δ_λ determined at $O(\lambda^2)$

 which is also $\sim \lambda^3 \log \Lambda^2 \log p^2$

It turns out that diagrams of this type cancel the nonlocal divergences. This actually works systematically to all orders in perturbation theory, a result known as the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) theorem.

In this lecture, I would like to demonstrate this explicitly for the 4-point function of ϕ^4 theory in 2-loop order. Along the way, I will introduce some useful technology for evaluating 2-loop diagrams.

First of all, I would like to define another prescription for ~~defining~~ the counterterms that is much simpler for higher-loop computations. The prescription on p. 1 was physically motivated and is called an "on-shell" scheme of renormalization. 't Hooft introduced a more mechanical prescription for renormalization that takes advantage of the explicit structure of dimensional regularization: He said, since divergent terms are characterized by poles in $\varepsilon = 4-d$, the "minimal subtraction" is to choose the counterterms exactly to subtract the poles. With this minimal subtraction, the mass and field strength normalization of the field are shifted in perturbation thy. But, if we account for this, the predictions of the thy relating physical quantities are the same as with on-shell renormalization.

To make this concrete, I will determine the counterterms for ϕ^4 with minimal subtraction. If we continue away from 4 dimensions, the coupling constant of ϕ^4 thy has mass dimension. It is convenient to keep λ as a dimensionless number by writing

$$\lambda \to \lambda (M^2)^{\varepsilon/2} \qquad \text{for} \quad d = 4-\varepsilon$$

In QED or QCD we would write $g^2 \to g^2 (M^2)^{\varepsilon/2}$.

Actually, it is conventional now to use the "modified minimal subtraction" prescription ($\overline{MS}$ or "em-ess-bar")

$$\lambda \to \lambda \left[M^2 e^{\gamma - \log 4\pi}\right]^{\varepsilon/2} \equiv \lambda (\bar{M}^2)^{\varepsilon/2}$$

With this prescription, the γ's and $\log 4\pi$'s of dimensional regularization cancel out to all orders in perturbation thy.

Begin with the 4-pt. function. To 1-loop order

$$+ \, \mathcal{O}(\lambda^3)$$

In dimensional regularization,

$$\uparrow P \;=\; \left(-i\left[\frac{\lambda(\bar M^2)^{\epsilon/2}}{2}\right]\right)^2 \cdot \int \frac{d^dk}{(2\pi)^d}\;\frac{i}{k^2-m^2}\;\frac{i}{(P+k)^2-m^2}$$

$$= \left(\frac{\lambda(\bar M^2)^{\epsilon/2}}{2}\right)^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{[k^2+x(1-x)P^2-m^2]^2}$$

$$= i\left(\frac{\lambda(\bar M^2)^{\epsilon/2}}{2}\right)^2 \int_0^1 dx\;\frac{1}{(4\pi)^{d/2}}\;\frac{\Gamma(2-d/2)}{[m^2-x(1-x)P^2]^{2-d/2}}$$

$$= \times \quad [-i\lambda(\bar M^2)^{\epsilon/2}]\cdot\left(-\frac{\lambda}{2}\int_0^1 dx\;\frac{1}{(4\pi)^2}\;\frac{\Gamma(\epsilon/2)(\bar M^2)^{\epsilon/2}}{[m^2-x(1-x)P^2]^{\epsilon/2}}\right)$$

now $\Gamma(z) = \frac{1}{z}\Gamma(1+z) = \frac{1}{z} - \gamma + O(z)$ as $z\to 0$

$$(\bar M^2)^{\epsilon/2} = \exp[\epsilon/2 \log \bar M^2] = 1 + \frac{\epsilon}{2}\log M^2 + O(\epsilon^2)$$

$$= [-i\lambda(\bar M^2)^{\epsilon/2}]\cdot\Big[-\frac{\lambda}{32\pi^2}\int_0^1 dx\,\Big\{\frac{2}{\epsilon} - \gamma + \log 4\pi + \log \bar M^2 - \log(m^2-x(1-x)P^2)\Big\} + O(\epsilon)\Big]$$

setting $\bar M^2 = M^2 e^{\gamma-\log 4\pi}$ cancels the factor $-\gamma+\log 4\pi$ and we find

$$= [-i\lambda(\bar M^2)^{\epsilon/2}]\left(-\frac{\lambda}{32\pi^2}\int_0^1 dx\left[\frac{2}{\epsilon} + \log\left(\frac{M^2}{m^2-x(1-x)P^2}\right)\right]\right).$$

adding all channels:

 $= \left[-i\lambda(\bar{M}^2)^{\epsilon/2}\right]\cdot\Big\{-\frac{3\lambda}{32\pi^2}\cdot\frac{2}{\epsilon}$

$$-\frac{\lambda}{32\pi^2}\int_0^1 dx\left[\log\frac{M^2}{m^2-x(1-x)s}+\log\frac{M^2}{m^2-x(1-x)t}+\log\frac{M^2}{m^2-x(1-x)u}\right]\Big\}$$

$+$ 

so if we set  $= -i\,\delta_\lambda\,(\bar{M}^2)^{\epsilon/2}$, $\delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}$

we remove the infinities and obtain a finite correction to the 4-pt. function. Similarly, for the 2-pt function

 $=$  $+$ 

 $= -i\,\frac{\lambda}{2}(\bar{M}^2)^{\epsilon/2}\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2}$

$$= \frac{\lambda}{2}(\bar{M}^2)^{\epsilon/2}\,\frac{-i}{(4\pi)^{d/2}}\,\frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$$

$$= -i\,\frac{\lambda\,(4\pi)^{\epsilon/2}}{2\cdot(4\pi)^2}\,\frac{(\bar{M}^2)^{\epsilon/2}}{(m^2)^{\epsilon/2}}\cdot m^2\cdot\frac{1}{1-d/2}\,\Gamma(2-d/2)$$

$$= -i\,\frac{\lambda m^2}{2(4\pi)^2}\,\Gamma(\epsilon/2)\,(4\pi)^{\epsilon/2}\left(\frac{\bar{M}^2}{m^2}\right)^{\epsilon/2}\cdot(-1)\,(1+\epsilon/2)+\cdots$$

near $\underline{\epsilon=0}$ $= -i\,\frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon}+\left(\log\frac{M^2}{m^2}+1\right)+O(\epsilon)\right]$

In $\overline{MS}$ we set $\delta_m = -\frac{\lambda m^2}{16\pi^2}\cdot\frac{1}{\epsilon}$ $\delta_Z = 0$, and we find a finite correction to the boson mass:

$$(m^2)_{\text{phys.}} = m^2 \left(1 + \frac{\lambda}{32\pi^2}\left(\log\frac{M^2}{m^2} + 1\right) + O(\lambda^2)\right)$$

"on-shell" mass $\qquad\qquad$ $\overline{MS}$ mass.

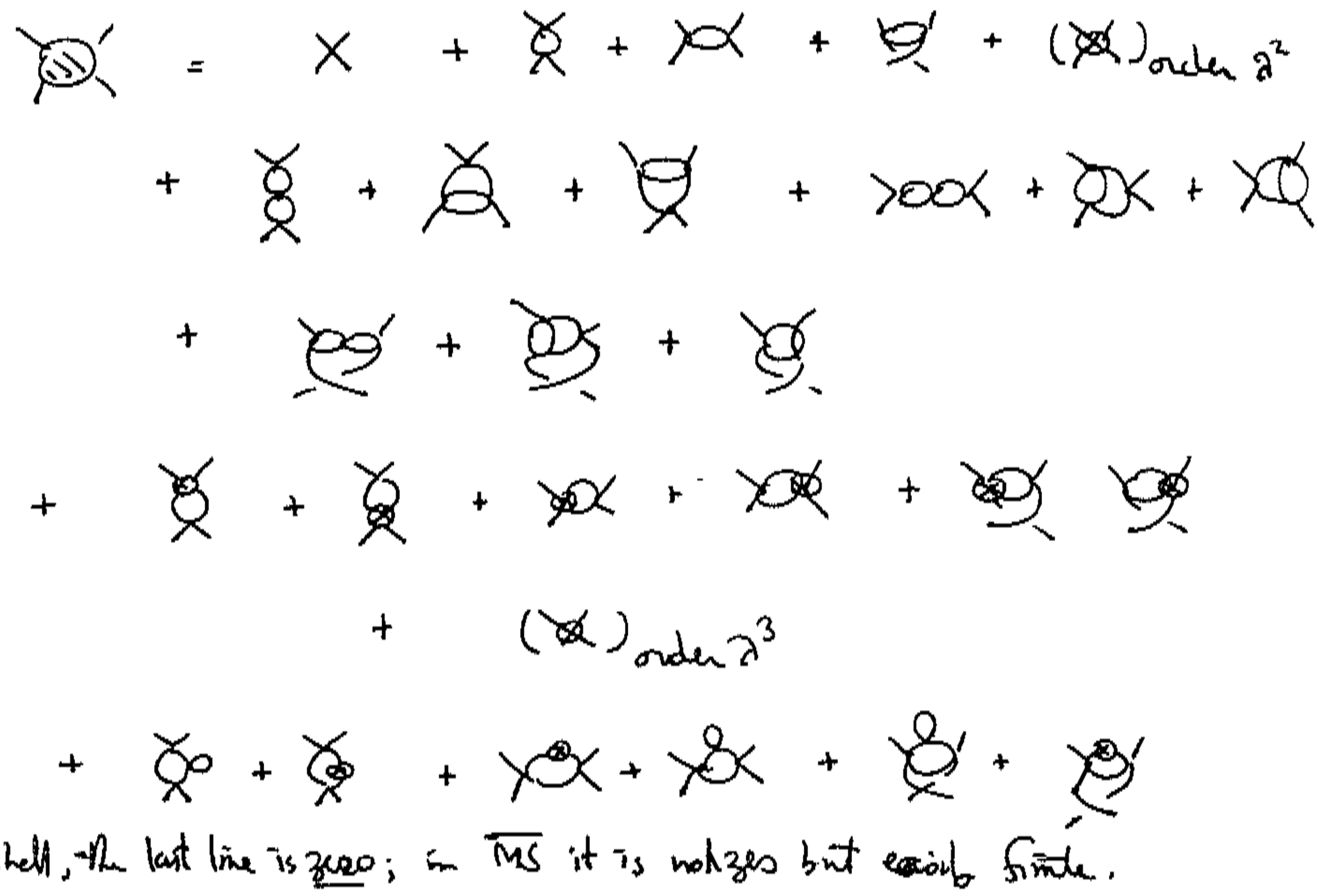
After correcting for the differences of masses and couplings, the predictions of the two schemes of renormalization are the same.

Once again, in $\overline{MS}$:

$$\delta_Z = 0 \qquad \delta_m = -\frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon} \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}$$

to 1-loop order.

Now, does this method continue to remove infinities to 2-loop order. To the next order,



In on-shell, the last line is zero; in $\overline{MS}$ it is nonzero but easily finite.

I claim that the divergences of the 2-loop s-channel diagrams

 +  + 

are cancelled by  +  , except for a divergent constant that can be absorbed by ( ) order λ^3.

A similar cancellation takes place for the t- and u-channel diagrams.

To show this, it is easy enough to evaluate  ,  , 

However, for  we will need some further tricks. I would therefore like to pause here and present some methods for evaluating 2-loop diagrams.

It turns out that the general 2-loop Feynman diagram with zero external momenta can be evaluated in a closed form. This was shown by van der Bij + Veltman. Those authors defined (after rotation to Euclidean space)

$$= \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \prod_i \frac{1}{p^2+M_i^2} \prod_j \frac{1}{q^2+M_j^2} \prod_k \frac{1}{(p+q)^2+M_k^2}$$

$$\equiv (M_1 \cdots M_i \cdots | \cdots M_j \cdots | \cdots M_k \cdots)$$

All of these integrals can be evaluated if we can evaluate

$$(M_1 | M_2 | M_3) = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{p^2+M_1^2} \frac{1}{q^2+M_2^2} \frac{1}{(p+q)^2+M_3^2}$$

For example

$$(M_{11}\ M_{12}\ |\ M_2\ |M_3) = \int\frac{d^dp}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\ \underbrace{\frac{1}{p^2+M_{11}^2}\ \frac{1}{p^2+M_{12}^2}}\ \frac{1}{q^2+M_2^2}\ \frac{1}{(p+q)^2+M_3^2}$$

$$= \frac{-1}{M_{11}^2-M_{12}^2}\left(\frac{1}{p^2+M_{11}^2} - \frac{1}{p^2+M_{12}^2}\right)$$

so

$$(M_{11}\ M_{12}\ |\ M_2\ |\ M_3) = \frac{-1}{M_{11}^2-M_{12}^2}\Big[(M_{11}\ (M_2|M_3) - (M_{12}|M_2|M_3)\Big]$$

Actually, $(M_1|M_2|M_3)$ is not the most convenient object, because it is divergent already at $d=3$. The integral

$$= (M_1\ M_1\ |M_2\ |\ M_3)$$

has 2-loops that are log divergent in $d=4$ but is not divergent for $d<4$. A simple formula allows us to compute $(M_1|M_2|M_3)$ in terms of this.

$$M_1^2\ (M_1 M_1\ |\ M_2\ |M_3) + M_2^2\ (M_2\ M_2\ |M_3|\ M_1) + M_3^2\ (M_3\ M_3\ |\ M_1\ \ |M_2)$$

$$= \int\frac{d^dp\ d^dq}{(2\pi)^{2d}}\left\{\frac{M_1^2}{(p^2+M_1^2)^2}\ \frac{1}{q^2+M_2^2}\ \frac{1}{(p+q)^2+M_3^2} + \quad 2 \text{ more terms}\right\}$$

$$= \int\frac{d^dp\,d^dq}{(2\pi)^{2d}}\left\{\left(\frac{1}{p^2+M_1^2} - \frac{p^2}{(p^2+M_1^2)^2}\right)\frac{1}{q^2+M_2^2}\ \frac{1}{(p+q)^2+M_3^2} + \cdots\right\}$$

we can write $\quad \dfrac{-p^2}{(p^2+M_1^2)^2} = \dfrac{\partial}{\partial\lambda}\ \dfrac{1}{(\lambda p^2+M_1^2)}\Big|_{\lambda=1}$, then

$$= \left(3 + \frac{\partial}{\partial\lambda}\right) \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{1}{\lambda p^2+M_1^2} \frac{1}{\lambda q^2+M_2^2} \frac{1}{\lambda(p+q)^2+M_3^2}$$

In the integral, change variables to $\lambda p^2 \to p^2 \quad \lambda q^2 \to q^2$

then $\quad d^dp\,d^dq \to \lambda^{-d}\, d^dp\,d^dq$ and we obtain

$$= (3-d) \int \frac{d^dp\, d^dq}{(2\pi)^{2d}} \frac{1}{p^2+M_1^2} \frac{1}{q^2+M_2^2} \frac{1}{(p+q)^2+M_3^2} = (3-d)(M_1|M_2|M_3)$$

so :

$$(M_1|M_2|M_3) = \frac{1}{3-d} \Big\{ M_1^2 (M_1 M_1 | M_2 | M_3) + M_2^2 (M_2 M_2 | M_3 | M_1) + M_3^2 (M_3 M_3 | M_1 | M_2) \Big\}$$

This formula explicitly indicates the pole in $(M_1|M_2|M_3)$ at $d=3$ and tells us how to analytically continue around it.

Now I would like to evaluate $(M_1 M_1 | M_2 | M_3)$

$$(M_1 M_1 | M_2 | M_3) = \int \frac{d^dp}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \frac{1}{(p^2+M_1^2)^2} \frac{1}{q^2+M_2^2} \frac{1}{(p+q)^2+M_3^2}$$

$$= \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+M_1^2)^2} \int_0^1 dx \int \frac{d^dq}{(2\pi)^d} \frac{1}{[x(q^2+M_2^2) + (1-x)((p+q)^2 + M_3^2)]^2}$$

$$= \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+M_1^2)^2} \frac{1}{[x(1-x)p^2 + xM_2^2 + (1-x)M_3^2]^{2-d/2}} \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}$$

$$= \frac{1}{(4\pi)^{d/2}} \Gamma(2-\tfrac{d}{2}) \int_0^1 dx\, [x(1-x)]^{-(2-\frac{d}{2})} \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+M_1^2)^2} \frac{1}{[p^2+\mu^2 M_1^2]^{2-\frac{d}{2}}}$$

where $\quad \mu^2 = \dfrac{xM_2^2 + (1-x)M_3^2}{x(1-x)M_1^2}$

now introduce another Feynman parameter, using the more general formula

$$\frac{1}{A^{\alpha} B^{\beta}} = \int_0^1 dy \, \frac{y^{\alpha-1}(1-y)^{\beta-1}}{[yA+(1-y)B]^{\alpha+\beta}} \, \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{1}{(4\pi)^{d/2}} \cancel{\Gamma(2-d/2)} \int_0^1 dx \, [x(1-x)]^{-\varepsilon/2} \, \frac{\Gamma(4-d/2)}{\Gamma(2)\cancel{\Gamma(2-d/2)}} \int_0^1 dy \int \frac{d^dp}{(2\pi)^d} \, \frac{(1-y)\, y^{1-d/2}}{[p^2+((1-y)+y\mu^2)M_1^2]^{4-d/2}}$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\cancel{\Gamma(4-d/2)}}{\underbrace{\Gamma(2)}_{=1}} \int_0^1 dx \, (x(1-x))^{-\varepsilon/2} \int_0^1 dy \, \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(4-d)}{\cancel{\Gamma(4-d/2)}} \, \frac{(1-y)\, y^{1-d/2}}{[((1-y)+y\mu^2)M_1^2]^{4-d}}$$

$$= \frac{1}{(4\pi)^{4-\varepsilon}} \Gamma(\varepsilon) \int_0^1 dx \, (x(1-x))^{-\varepsilon/2} \int_0^1 dy \, \frac{(1-y)\, y^{\varepsilon/2-1}}{[(1-y)+y\mu^2]^{\varepsilon}} \, (M_1^2)^{-\varepsilon}$$

Since $\Gamma(\varepsilon) \sim \frac{1}{\varepsilon}$, one divergence is apparent here. However, the diagram

 $\sim$  $3\, g\lambda^2 \sim g^2\lambda^2$

so we expect a $\frac{1}{\varepsilon^2}$ divergence. The second $\frac{1}{\varepsilon}$ comes from the y integral:

$$\int_0^1 dy \, y^{\varepsilon/2-1} = \frac{2}{\epsilon}$$

To make it more explicit how to reduce the integral, integrate by parts: $y^{\varepsilon/2-1} = \frac{2}{\epsilon}\frac{d}{dy} y^{\varepsilon/2}$ so

$$\int_0^1 dy \, \frac{y^{\varepsilon/2-1}(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}} = \underbrace{\frac{2}{\epsilon} \frac{y^{\varepsilon/2}(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}}\Big|_0^1}_{=0 \text{ for } \varepsilon>0} - \int_0^1 dy \, \frac{2}{\epsilon} y^{\varepsilon/2} \frac{d}{dy}\left(\frac{(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}}\right)$$

Finally then, we find:

$$(M_1 M_1 | M_2 | M_3) = \frac{1}{(4\pi)^4}\,\frac{2}{\epsilon^2}\,\Gamma(1+\epsilon)\,(M_1^2)^{-\epsilon}\,(4\pi)^{\epsilon}\cdot$$

$$\cdot \int_0^1 dx\,[x(1-x)]^{-\epsilon/2} \int_0^1 dy\; y^{\epsilon/2}\left(-\frac{d}{dy}\right)\left[\frac{(1-y)}{[(1-y)+y\mu^2]^{\epsilon}}\right]$$

This expression can be expanded about $\varepsilon = 0$. Consider first the y integral. Since the prefactor is $\sim \frac{1}{\varepsilon^2}$, we need to expand only to $\mathcal{O}(\varepsilon^2)$. So write

$$y^{\varepsilon/2} = 1 + \frac{\varepsilon}{2}\log y - \frac{\varepsilon^2}{8}\log^2 y + \mathcal{O}(\varepsilon^3)$$

Then the first term of the y integral is:

$$\int_0^1 dy \cdot 1 \cdot \left(-\frac{d}{dy}\right)\left(\frac{(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}}\right) = -\frac{(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}}\Bigg|_0^1 = +1$$

The third term is also easy:

$$\int_0^1 dy\left(-\frac{\varepsilon^2}{8}\log^2 y\right)\left(-\frac{d}{dy}\right)\left(\frac{(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}}\right) = \int_0^1 dy\, -\frac{\varepsilon^2}{8}\log^2 y\cdot[1+\mathcal{O}(\varepsilon)]$$

$$= +\frac{\varepsilon^2}{4} \qquad \text{since} \qquad \int_0^1 dy\,\log^2 y = +2$$

For the second term, write

$$-\frac{d}{dy}\left(\frac{(1-y)}{[(1-y)+y\mu^2]^{\varepsilon}}\right) = -\frac{d}{dy}\left[(1-y) + (-\varepsilon)(1-y)\log[(1-y)+y\mu^2] + \mathcal{O}(\varepsilon^2)\right]$$

$$= 1 + \varepsilon\,\frac{d}{dy}\left[(1-y)\log(1-(1-\mu^2)y)\right] + \cdots$$

Now $\int_0^1 dy\, \frac{\varepsilon}{2}\log y \cdot 1 = -\frac{\varepsilon}{2}$ so the hard part is

$$\int_0^1 dy\left(\frac{\varepsilon}{2}\log y\right)\cdot \varepsilon\,\frac{d}{dy}\left[(1-y)\log(1-(1-\mu^2)y)\right]$$

$$= \frac{\varepsilon^2}{2}\,\underbrace{(\log y)\,(1-y)\,\log(1-(1-\mu^2)y)\Big|_0^1}_{=0 \text{ since } \log(1-(1-\mu^2)y)\,\sim\, y(1-\mu^2) \text{ as } y\to 0} - \frac{\varepsilon^2}{2}\int_0^1 \frac{dy}{y}(1-y)\log(1-(1-\mu^2)y)$$

$$= -\varepsilon^2/2 \underline{\int_0^1 dy\, \frac{1}{y} \log(1-(1-\mu^2)y)} + \varepsilon^2/2 \underbrace{\int_0^1 dy \log(1-(1-\mu^2)y)}_{= \frac{1}{(1-\mu^2)}[-\mu^2\log\mu^2 - (1-\mu^2)]}$$

This term cannot be reduced further.
It is a special function, one which appears
frequently in Feynman diagram computations: the "dilogarithm"

$$Li_2(z) = -\int_0^1 dy\, \frac{1}{y} \log(1-zy)$$

$Li_2(z)$ satisfies:

$$Li_2(0) = 0$$

$$Li_2(1) = \int_0^1 \frac{dy}{y} \sum_{n=1}^{\infty} \frac{(zy)^2}{n}\Big|_{z=1} = \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$Li_2(-1) = \sum_1^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

assembling all of the pieces:

$$(M_1 M_1 | M_2 | M_3) = \frac{1}{(4\pi)^4} \frac{2}{\epsilon^2} \Gamma(1+\varepsilon)(M_1^2)^{-\varepsilon}(4\pi)^{\varepsilon}$$

$$\int_0^1 dx\, (x(1-x))^{-\varepsilon/2} \left[1 - \varepsilon/2 + \varepsilon^2/4 + \varepsilon^2/2 \left(Li_2(1-\mu^2) - \frac{\mu^2 \log\mu^2}{1-\mu^2} - 1 \right) \right]$$

Next. $\displaystyle\int_0^1 dx\, [x(1-x)]^{-\varepsilon/2} = \frac{[\Gamma(1-\varepsilon/2)]^2}{\Gamma(2-\varepsilon)}$ except that for the ε^2 terms we can set $[x(1-x)]^{-\varepsilon/2} \approx 1$

$$= \frac{1}{(4\pi)^4} \frac{2}{\epsilon^2} \Gamma(1+\epsilon)(4\pi)^{\epsilon}(M_1^2)^{-\varepsilon} \cdot \frac{(\Gamma(1-\varepsilon/2))^2}{\Gamma(2-\varepsilon)}$$

$$\cdot \left(1 - \varepsilon/2 - \varepsilon^2/4 + \varepsilon^2/2 \int_0^1 dx \left(Li_2(1-\mu^2) - \frac{\mu^2\log\mu^2}{1-\mu^2} \right) \right)$$

Finally. $\displaystyle\frac{(\Gamma(1+z))^2}{\Gamma(2+2z)} = 1 - 2z + (4 - \pi^2/6) z^2 + \cdots$

$$\frac{[\Gamma(1-\varepsilon/2)]^2}{\Gamma(2-\varepsilon)} = 1 + \varepsilon + \varepsilon^2(1 - \pi^2/24) + \cdots$$

so that

$$(M_1 M_1 | M_2 | M_3) = \frac{1}{(4\pi)^4} \Gamma(1+\varepsilon) \left(\frac{M_1^2}{4\pi}\right)^{-\varepsilon}$$

$$\cdot \left\{ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} + \frac{1}{2} - \pi^2/12 \right.$$

$$\left. + \int_0^1 dx \left(Li_2(1-\mu^2) - \frac{\mu^2 \log \mu^2}{1-\mu^2} \right) \right\}$$

where $\mu^2 = \left[\frac{x M_2^2 + (1-x) M_3^2}{x(1-x) M_1^2} \right]$

This is a 1-parameter integral representation of $(M_1 | M_2 | M_3)$ that is already quite useful. van der Bij and Veltman showed that the $\int dx$ can be done explicitly in terms of dilogarithms. Often, some pairs of $M_1 M_2 M_3$ are equal or zero, leading to simpler formulae. For example,

$$(M M | 0 | M_3) = \frac{\Gamma(1+\varepsilon)}{(4\pi)^{4}} \left\{ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} (1 - 2 \log M_3^2) \right.$$

$$+ \frac{1}{2} - \pi^2/12 - Li_2 \left(1 - \frac{M^2}{M_3^2}\right)$$

$$+ \frac{1}{2} \log^2 M^2 - \frac{1}{2} \log^2 M_3^2 + \log M_3^2 \log M^2 - \log M^2$$

$$\left. + O(\varepsilon) \right\}$$

For more such results, see:

van der Bij + Veltman Nucl. Phys. B231, 205 (1984)
Martin, hep-ph/9608224

in the Appendices, of course!

With this preparation, let's go back to the claim made on p. 8:

The divergences of  +  +  are cancelled by  ,
except for a local divergence that can be absorbed into () order λ^3.
In the remainder of the lecture, I will show this by explicit calculation.

Using the $\overline{MS}$ counterterms on p. 7:

$P\uparrow$  +  $= (-i\lambda)\left(-i\,\frac{3\lambda^2}{16\pi^2}\,\frac{1}{\epsilon}\right)\left[(\bar{M}^2)^{+\epsilon/2}\right]^2 \cdot \frac{1}{2}\cdot 2$

$$\cdot \int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2}\,\frac{i}{(p+k)^2-m^2}$$

$$= (\bar{M}^2)^{+\epsilon/2}\cdot \frac{3\lambda^3}{16\pi^2}\,\frac{1}{\epsilon}\int_0^1 dz\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-\tfrac{d}{2})\,\frac{(\bar{M}^2)^{\epsilon/2}}{[m^2-z(1-z)P^2]^{2-d/2}}$$

$$= i(\bar{M}^2)^{\epsilon/2}\cdot\frac{3\lambda^3}{(4\pi)^4}\int_0^1 dz\,\frac{1}{\epsilon}\left\{\frac{2}{\epsilon}+\log\frac{M^2}{m^2-z(1-z)P^2} + O(\epsilon)\right\}$$

actually, the $\frac{1}{\epsilon}\cdot O(\epsilon)$ term gives a finite contribution, but here we are only interested in the divergent terms.

$$= i(\bar{M}^2)^{\epsilon/2}\cdot\frac{3\lambda^3}{(4\pi)^4}\cdot\left\{\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\int_0^1 dz\,\log\frac{M^2}{m^2-z(1-z)P^2} + (\text{finite})\right\}$$

Next. $P\uparrow$  $= \left[-i\lambda(\bar{M}^2)^{\epsilon/2}\right]^3\left(\frac{1}{2}\right)^2\cdot\left[\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2}\,\frac{i}{(p+k)^2-m^2}\right]^2$

$$= \frac{1}{4} \cdot i \lambda^3 (\bar{M}^2)^{\epsilon/2} (\bar{M}^2)^{\epsilon} \left(\frac{-i}{(4\pi)^{d/2}} \int_0^1 dz \, \frac{\Gamma(2-d/2)}{[m^2 - z(1-z)P^2]^{2-d/2}} \right)^2$$

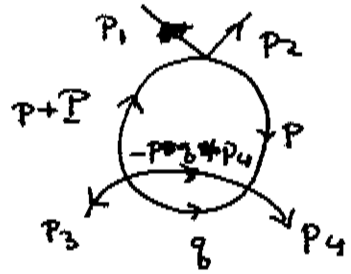
$$= -i \frac{\lambda^3}{4} (\bar{M}^2)^{\epsilon/2} \left[\frac{2}{\epsilon} + \int_0^1 dz \, \log \frac{M^2}{m^2 - z(1-z)P^2} \right]^2 \left(\frac{1}{(4\pi)^2} \right)^2$$

$$= -i (\bar{M}^2)^{\epsilon/2} \cdot \frac{\lambda^3}{4(4\pi)^4} \left(\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \int_0^1 dz \, \log \frac{M^2}{m^2 - z(1-z)P^2} + \text{finite} \right)$$

Now we need to evaluate

$P \uparrow$  with nonzero external momenta.

Route the momenta as follows:



$P = (p_1 + p_2)$

$$= \left[-i \lambda (\bar{M}^2)^{\epsilon/2} \right]^3 \int \frac{d^dp \, d^dq}{(2\pi)^{2d}} \frac{i}{(p+P)^2 - m^2} \frac{i}{p^2 - m^2} \frac{i}{q^2 - m^2} \frac{i}{(p+q-p_4)^2 - m^2} \cdot \frac{1}{2}$$

$$= \frac{i \lambda^3 (\bar{M}^2)^{3\epsilon/2}}{2} \int \frac{d^dp \, d^dq}{(2\pi)^{2d}} \int_0^1 dz \frac{1}{[z(p+P)^2 + (1-z)p^2 - m^2]^2} \int_0^1 dx \frac{1}{[x(p+q-p_4)^2 + (1-x)q^2 - m^2]^2}$$

$$= \frac{i \lambda^3 (\bar{M}^2)^{3\epsilon/2}}{2} \int \frac{d^dp}{(2\pi)^d} \int_0^1 dz \int_0^1 dx \frac{1}{[p^2 + 2z p \cdot P + z P^2 - m^2]^2}$$

$$\cdot \frac{i}{(4\pi)^{d/2}} \Gamma(2 - d/2) \frac{1}{[m^2 - x(1-x)(p-p_4)^2]^{2-d/2}}$$

$$= \frac{i\lambda^3(\bar{M}^2)^{3\epsilon/2}}{2} \int \frac{d^dp}{(2\pi)^d} \int_0^1 dz \int_0^1 dx \; [x(1-x)]^{-\epsilon/2} \; \frac{i}{(4\pi)^{d/2}} \Gamma(2-d/2)$$

$$\cdot \frac{1}{\left[\frac{m^2}{x(1-x)} - (p-p_4)^2\right]^{\epsilon/2}} \; \frac{1}{\left[m^2 - (p^2 + 2zp\cdot P + zP^2)\right]^2}$$

Let's rotate all momenta to Euclidean space; we can come back when we are done:

$$= \frac{i\lambda^3(\bar{M}^2)^{3\epsilon/2}}{2} \; \frac{i}{(4\pi)^{d/2}} \Gamma(\epsilon/2) \cdot i \int dz \int_0^1 dx \; [x(1-x)]^{-\epsilon/2}$$

$$\frac{1}{\left[m^2/x(1-x) + (p^2 - 2p\cdot p_4 + p_4^2)\right]^{\epsilon/2} \left[m^2 + p^2 + 2zp\cdot P + zP^2\right]^2}$$

as on p.11, combine these denominators:

$$= -i \, \frac{\lambda^3(\bar{M}^2)^{3\epsilon/2}}{2} \; \frac{\cancel{\Gamma(\epsilon/2)}}{(4\pi)^{d/2}} \int_0^1 dz \int_0^1 dx \; [x(1-x)]^{-\epsilon/2} \int_0^1 dy \; (1-y) \, y^{-1+\epsilon/2}$$

$$\cdot \frac{\Gamma(2+\epsilon/2)}{\cancel{\Gamma(\epsilon/2)}\,\Gamma(2)} \int \frac{d^dp}{(2\pi)^d} \frac{1}{\left[\begin{array}{c} p^2 + 2p\cdot((1-y)zP - yp_4) + (1-y)zP^2 + yp_4^2 \\ + m^2\left(y\frac{1}{x(1-x)} + (1-y)\right) \end{array}\right]^{\overset{2+\epsilon/2}{4-d/2}}}$$

the denominator is

Let $k = p + (1-y)zP - yp_4$ then

$$\text{Denom} = k^2 + (1-y)zP^2 + yp_4^2 - (1-y)^2z^2P^2 + 2y(1-y)zP\cdot p_4 - y^2p_4^2$$
$$+ m^2\left(1 + y\left[\frac{1}{x(1-x)} - 1\right]\right)$$

$$= k^2 + y(1-y)z\underbrace{(P+p_4)^2}_{= -p_3} + (1-y)^2 z(1-z)P^2 + y(1-y)(1-z)p_4^2$$
$$+ m^2\left(1 + y\left[\frac{1}{x(1-x)} - 1\right]\right)$$

$$= \quad -i \, \frac{\lambda^3 (\bar{M}^2)^{3\varepsilon/2}}{2} \, \frac{1}{(4\pi)^{d/2}} \, \frac{\cancel{\Gamma(2+\varepsilon/2)}}{\underbrace{\Gamma(2)}_{1}} \cdot \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(4-d)}{\cancel{\Gamma(4-d/2)}}$$

$$\cdot \int_0^1 dz \int_0^1 dx \, [x(1-x)]^{-\varepsilon/2} \int_0^1 dy \, (1-y)\, y^{\varepsilon/2 - 1} \, \frac{1}{\left[\begin{array}{c}(1-y)^2 z(1-z) P^2 + y(1-y) z \, p_3^2 + y(1-y)(1-z) p_4^2 \\ + m^2 (1 + y(\;\;))\end{array}\right]^{\varepsilon}}$$

call this $\Delta(y, z, x)$

as on p. 11, look at the dy integral. Write $y^{\varepsilon/2-1} = \frac{2}{\epsilon} \frac{d}{dy} y^{\varepsilon/2}$ and integrate by parts:

$$\int dy \, (1-y) \, y^{\varepsilon/2-1} \frac{1}{\Delta^\varepsilon} = \underbrace{\frac{2}{\epsilon} \frac{y^{\varepsilon/2}(1-y)}{\Delta^\varepsilon}\Big|_0^1}_{=\,0} - \int_0^1 dy \, \frac{2}{\epsilon} \, y^{\varepsilon/2} \frac{d}{dy}\left(\frac{(1-y)}{[\Delta]^\varepsilon}\right)$$

$$\Gamma(4-d) = \Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon) \quad , \text{so}$$

$$= \quad -i \, \lambda^3 (\bar{M}^2)^{\varepsilon/2} \cdot \frac{1}{2} \cdot \frac{(\bar{M}^2)^\varepsilon}{(4\pi)^d} \, \Gamma(1+\varepsilon) \cdot \frac{2}{\epsilon^2}$$

$$\int_0^1 dz \int_0^1 dx \, (x(1-x))^{-\varepsilon/2} \int_0^1 dy \; y^{\varepsilon/2} \left(-\frac{d}{dy}\left[\frac{(1-y)}{\Delta^\varepsilon}\right]\right)$$

For this lecture, I am not interested in the <u>finite</u> terms in this expression, but I would like to extract the terms of order $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$. So, the dy integral can be approximated:

$$\int_0^1 dy \, \left(1 + \tfrac{\varepsilon}{2} \log y + \cdots\right) \left[-\frac{d}{dy}\left(\frac{(1-y)}{\Delta^\varepsilon}\right)\right]$$

$$= \int dy \left(-\frac{d}{dy}\frac{(1-y)}{\Delta^\varepsilon}\right) + \frac{\varepsilon}{2} \int_0^1 dy \, \log y \cdot 1 + \mathcal{O}(\varepsilon) + \cdots$$

$$= \frac{-(1-y)}{\Delta^\varepsilon}\Big|_0^1 + \frac{\varepsilon}{2}(-1) + \cdots = \frac{1}{[\Delta(y=0, z, x)]^\varepsilon} - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)$$

so


$$\Big|_{\text{Euclid. chrom}} = -i\, \lambda^3 (\bar{M}^2)^{\varepsilon/2} \cdot \frac{(\bar{M}^2)^{\varepsilon}}{(4\pi)^{4-\varepsilon}} \Gamma(1+\varepsilon) \cdot \frac{1}{\epsilon^2}$$

$$\cdot \int_0^1 dz \int_0^1 dx\, [x(1-x)]^{-\varepsilon/2} \left[1 - \varepsilon/2 - \varepsilon \lg [\Delta(y=0,z)] + O(\varepsilon^2) \right]$$

now $\Delta(y=0, z, x) = z(1-z) P^2 + m^2$ (indep of x!)

we can now rotate back to Minkowski space $\rightarrow (m^2 - z(1-z)P^2)$

$$\int_0^1 dx\, [x(1-x)]^{-\varepsilon/2} = \frac{[\Gamma(1-\varepsilon/2)]^2}{\Gamma(2-\varepsilon)} = 1 + \varepsilon + O(\varepsilon^2)$$

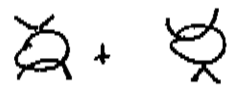
$$= -i\, \lambda^3 \frac{(\bar{M}^2)^{\varepsilon/2}}{(4\pi)^4} \cdot \frac{1}{\epsilon^2} \left(1 + \varepsilon/2 + \varepsilon \int_0^1 dz \lg\left(\frac{M^2}{m^2 - z(1-z)P^2}\right) + \dots\right)$$

$$= -i\, \lambda^3 \frac{(\bar{M}^2)^{\varepsilon/2}}{(4\pi)^4} \left(\frac{1}{\epsilon^2} + \frac{1}{2}\frac{1}{\epsilon} + \frac{1}{\epsilon}\int_0^1 dz \lg\left(\frac{M^2}{m^2 - z(1-z)P^2}\right) + \dots\right)$$

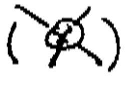
Excellent! We have now computed:


$$= i\, (\bar{M}^2)^{\varepsilon/2} \frac{\lambda^3}{(4\pi)^4} \cdot \left(\frac{6}{\epsilon^2} + \frac{3}{\epsilon}\int_0^1 dz \lg \frac{M^2}{m^2 - z(1-z)P^2}\right)$$


$$= i(\bar{M}^2)^{\varepsilon/2} \frac{\lambda^3}{(4\pi)^4} \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\int_0^1 dz \lg \frac{M^2}{m^2 - z(1-z)P^2}\right)$$


$$= i(\bar{M}^2)^{\varepsilon/2} \frac{\lambda^3}{(4\pi)^4} \left(-\frac{2}{\epsilon^2} - \frac{1}{\epsilon} - \frac{2}{\epsilon}\int_0^1 dz \lg \frac{M^2}{m^2 - z(1-z)P^2}\right)$$

the nonlocal divergence cancels, as promised. The divergences are now all cancelled if we set


$$(\ )_{\lambda^3} = -i\,(\bar{M}^2)^{\varepsilon/2}(\delta_\lambda) \qquad \delta_\lambda = \frac{\lambda^3}{(4\pi)^4}\left(\frac{3}{\epsilon^2} - \frac{1}{\epsilon}\right)$$

plus t and u channel contributions.

Again, the divergences of  are cancelled through the 2-loop level if we set

 $= -i(\bar{M}^2)^{\epsilon/2}\,\delta_\lambda$

$$\delta_\lambda = \frac{\lambda^2}{(4\pi)^2}\cdot\frac{3}{\epsilon} + \frac{\lambda^3}{(4\pi)^4}\cdot\left(\frac{9}{\epsilon^2} - \frac{3}{\epsilon}\right) + O(\lambda^4)$$

This illustrates a cancellation of divergences that actually goes through to all orders.