Lattice Models of Scalar Fields and Gauge Fields

In the previous lecture, we discussed the systematics of scalar field theories as a function of their dimensionality. In the process, we introduced the nonlinear sigma model. This model is closely related to statistical mechanics models of magnetic systems in crystal lattices.

Let's write these models explicitly in the lattice setting. Let i be points of a d-dimensional (hyper)-cubic lattice, $\hat{a}$ the elementary vectors $\hat{1}, \hat{2} \cdots$. Then a spin effective Hamiltonian would be:

$$\mathcal{H} = -J \sum_{i\hat{a}} \vec{S}_i \cdot \vec{S}_{i+a}$$

where $\vec{S}$ is an N-component unit vector. $N=1$, $S = \pm 1$, is called the Ising model. For $N>1$, we obtain a model similar to the nonlinear sigma model. The partition function is

$$Z = \prod_i \int d\vec{S}_i \; e^{+\beta J \sum_{ia} \vec{S}_i \cdot \vec{S}_{i+a}} \qquad \beta = \frac{1}{T}$$

If we take the limit $\beta \to \infty$ $(T \to 0)$, nearest-neighbor spins want to be aligned. We can then approximate $\vec{S}_i$ as a continuum field $\vec{n}(x_i)$, $|\vec{n}|^2 = 1$ as in the nonlinear sigma model.

$$\vec{S}_i \cdot \vec{S}_{i+a} \cong |\vec{n}(x_i)|^2 + \vec{n}(x_i)\, \hat{a} \cdot \partial \vec{n} + \tfrac{1}{2} \vec{n}\, \hat{a}\partial \hat{a} \cdot \partial \vec{n} + \cdots$$

summing over directions $\hat{\alpha}$

$$\sum_{i\alpha} \vec{S}_i \cdot \vec{S}_{i+\alpha} \;\cong\; \sum_i \tfrac{1}{2}\vec{n} \cdot \partial^2 \vec{n} \;\cong\; -\int d^d x \; \tfrac{1}{2}\partial_\mu \vec{n} \cdot \partial^\mu \vec{n}$$

in d-dimensional Euclidean space. So, as $T \to 0$, the spin model becomes a lattice regularization of the continuum nonlinear sigma model.

An advantage of working with lattice models is that we can apply approximate schemes with no obvious analogue in the continuum. The magnetic phase transition in a spin system is often studied in mean field theory: Isolate one spin and consider it to interact with the average spin orientation of its neighbors:

$$Z \approx \Big[\int d\vec{s} \;\; e^{\beta J (2d) \cdot \vec{s} \cdot \langle \vec{s} \rangle} \Big]^N \leftarrow \text{\# of lattice sites}$$

$2d$ ↑ 2d neighbors; $\langle \vec{s} \rangle$ ← avg value of $\vec{s}$ for the neighbors

We solve for $\langle \vec{s} \rangle$ self-consistently:

$$\langle \vec{s} \rangle = \frac{\int d\vec{s} \;\; e^{2d\beta J \langle \vec{s} \rangle \cdot \vec{s}} \;\; \vec{s}}{\int d\vec{s} \;\; e^{2d\beta J \langle \vec{s} \rangle \cdot \vec{s}}}$$

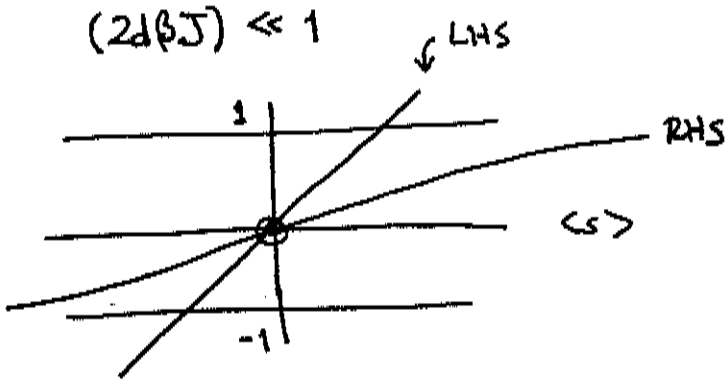
Let's work this out explicitly for Ising spins $s = \pm 1$

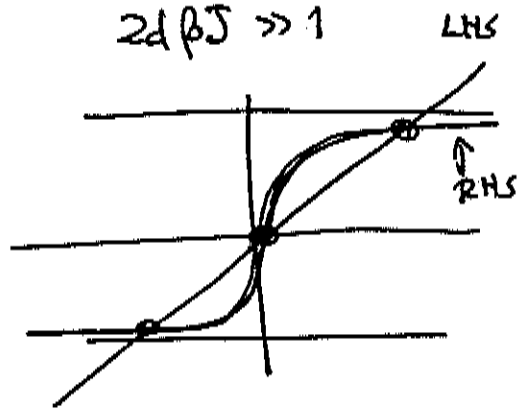
$$\langle s \rangle = \frac{e^{2d\beta J \langle s \rangle \cdot (+1)} \cdot (+1) + e^{2d\beta J (-1)} (-1)}{e^{2d\beta J \langle s \rangle (+1)} + e^{\beta d \beta J \cdot (-1)}}$$

that is

$$\langle s\rangle = \tanh(2d\beta J\langle s\rangle)$$

We can solve this equation graphically:





For $\beta \to 0$, $T \to \infty$, there is only one solution $\langle s\rangle = 0$.

For $\beta \to \infty$, $T \to 0$, there are three solutions. One can check that the solution $\langle s\rangle = 0$ has higher free energy that the solutions $|\langle s\rangle| \neq 0$

As $\beta \to +\infty$ $\langle s\rangle \to \pm 1$.

Thus, we have a magnetized state at low T and a disordered state at high T. The phase transition between these states occurs where the two curves are tangent at $\langle s\rangle = 0$,

$$2d\beta_c J = 1 \quad \text{or} \quad T_c = 2dJ$$

A similar analysis can be done for any N. It is convenient to normalize the integral $\int d\vec{s}$ over the unit sphere so that $\int d\vec{s} = 1$

Then
$$\int d\vec{s} = 1 \ , \quad \int d\vec{s}\ s^i = 0 \ , \quad \int d\vec{s}\ s^i s^j = \frac{\delta^{ij}}{N}$$

[check on the last result : set $i=j$ & sum $\int d\vec{s}\ |\vec{s}|^2 = 1$.]

Again we write
$$\langle s^i \rangle = \frac{\int d\vec{s}\ e^{2d\beta J \langle \vec{s}\rangle \cdot \vec{s}}\ s^i}{\int d\vec{s}\ e^{2d\beta J \langle \vec{s}\rangle \cdot \vec{s}}}$$

as $\beta \to 0$ $\langle s^i \rangle = 0$. For $\beta \to \infty$ $|\langle s^i \rangle| \to 1$. For small $\langle s \rangle$:
$$\int d\vec{s}\ e^{2d\beta J \langle \vec{s}\rangle \cdot \vec{s}} \cong 1 + \mathcal{O}(\beta^2)$$
$$\int d\vec{s}\ e^{2d\beta J \langle s \rangle \cdot \vec{s}}\ s^i = \int ds\ (1 + 2d\beta J \langle \vec{s} \rangle \cdot \vec{s} + \cdots)\ s^i$$
$$= 2d\beta J \frac{1}{N} \langle s^i \rangle$$

so
$$\langle s^i \rangle = \frac{2d\beta J}{N} \langle s^i \rangle + \cdots$$

so the curves are tangent, and the phase transition occurs, at
$$\frac{2d\beta_c J}{N} = 1 \qquad \text{or} \qquad T_c = \frac{2dJ}{N}$$

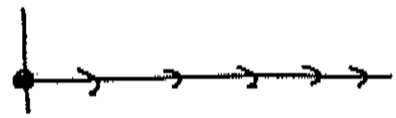
From the exact solution for $N \to \infty$ given the previous lecture, we know ~~that~~ there is a phase transition for $d > 2$, but for $d = 2$ the disordered phase extends from $T = \infty$ to $T = 0$.

We have a few more pieces of information:

① At $d=2$, the Mermin-Wagner theorem forbids the magnetic order $\langle \vec{s} \rangle \neq 0$; see Peskin + Schroeder Problem 11.1

for a physical argument.

② For $N > 2$, the continuum nonlinear sigma model is asymptotically free for β large. So it is suggested that for all $N > 2$ there is a renormalization group flow:



$T = \frac{1}{\beta}$

toward the disordered region at large T.

For $N=1$, the Ising model, there is a magnetic phase transition at $d=2$, but there is no phase transition at $d=1$. For $N=2$, in $d=2$ there is a phase transition at finite T, but $\langle s \rangle = 0$ in both phases. The low-temperature phase is called the "Kosterlitz-Thouless phase". It has power-law decay of spin correlations.

In all cases, we have a disordered phase at sufficiently high T. This is not an accident. It can be shown that all of these spin models have well-defined perturbation expansions for high T. These perturbation series, unlike those from Feynman diagrams, are actually convergent in a region $T > T_*$

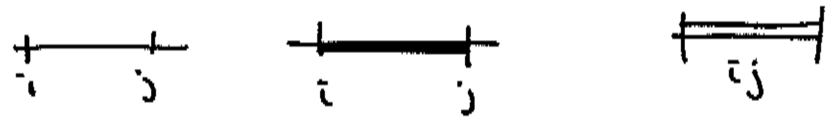
To develop the perturbation they write

$$Z = \prod_i \int d\vec{s}_i \; e^{\beta J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j} \qquad , \quad \langle ij \rangle = i \neq j \text{ nearest neighbors.}$$

$$= \prod_i \int d\vec{s}_i \prod_{\langle ij \rangle} \left(1 + \beta J \, \vec{s}_i \cdot \vec{s}_j + \tfrac{1}{2} [\beta J (\vec{s}_i \cdot \vec{s}_j)]^2 + \cdots \right)$$

Now integrate over the $\vec{s}_i$. Only those terms are nonzero for which we have an even number of powers of s_i at each site. We can represent the nonzero terms diagrammatically. On the bond $\langle ij \rangle$, represent:

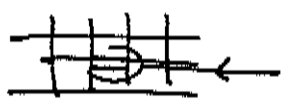
$$1 \quad + \quad \beta J\, \vec{s}_i \cdot \vec{s}_j \quad + \quad \frac{1}{2}\left(\beta J\, \vec{s}_i \cdot \vec{s}_j\right)^2 + \cdots$$



Then the first nonzero terms in a power series in β are:

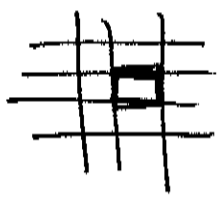
$= 1$

(empty)



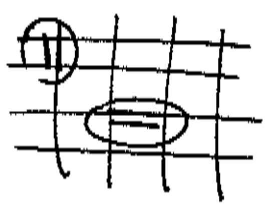
$$dN \cdot \frac{1}{2}(\beta J)^2 \cdot \frac{1}{N}$$

↑ # of sites



$$N \cdot \frac{d(d-1)}{2} \cdot (\beta J)^4 \cdot \left(\frac{1}{N}\right)^4 \cdot N$$

there are other terms of order β^4:



$$\sim \frac{1}{2}\left(\frac{dN}{2}(\beta J)^2 \frac{1}{N}\right)^2 + O(N)$$

There is an exponentiation of disconnected diagrams.

$$Z = \exp\left[N \left\{ \frac{d}{2N}(\beta J)^2 + \frac{d(d-1)}{2}(\beta J)^4 \frac{1}{N^3} + \cdots \right\} \right]$$

$= \quad + \quad \square \quad + \cdots$

so $\quad Z = e^{-\beta F}$, with $F = Nf$ (extensive)

and f is given by a power series in $\beta = \frac{1}{T}$.

It is instructive to compute the spin correlation function $\langle \vec{S}_0 \cdot \vec{S}_I \rangle$ in this perturbation series. The first nonzero term is

$\langle \vec{S}_0 \cdot \vec{S}_I \rangle =$

0 $\qquad\qquad\qquad$ I

$$= \left(\frac{\beta J}{N}\right)^I \cdot N \quad (1 + O(\beta^2))$$

$$\cong \quad N \cdot \exp\left[-I \cdot \left(\log \frac{N}{\beta J}\right)\right]$$

this is of the form

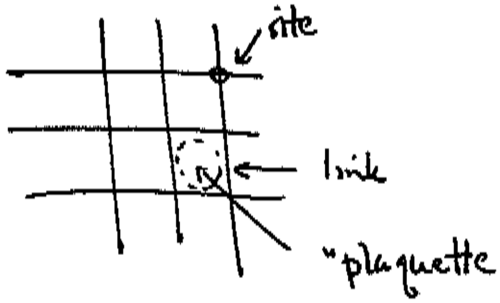
$$\langle \vec{S}(0) \cdot \vec{S}(x) \rangle \sim e^{-|\vec{x}|/\xi}$$

where $\quad \xi \sim \left(\log \frac{N}{\beta J}\right)^{-1} \to 0$ as $T \to \infty$

We find exponentially decaying correlations, with $\xi \to 0$ as $T \to \infty$, the characteristic of the disordered phase.

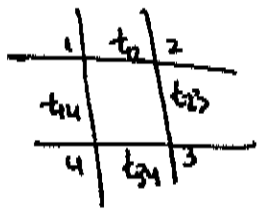
All of the models we have discussed so far have global symmetries only. Let's now construct lattice models with <u>local</u> symmetry.

Consider first the Ising case. Put element $t_{ij} = \pm 1$ on the links of a cubic lattice



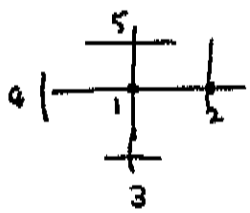
set $$\mathcal{H} = - \sum_{\text{plaquettes}} [tttt]_p .$$

where $tttt$ is the product of the four t_{ij}'s around a plaquette



$$[tttt] = (t_{12} t_{23} t_{34} t_{41})$$

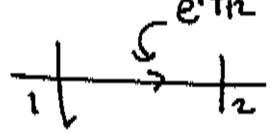
Under the Z_2 symmetry on the site 1



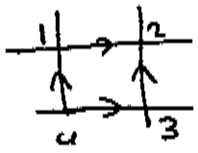
$$t_{12} \to -t_{12}$$
$$t_{13} \to -t_{13}$$
$$t_{14} \to -t_{14}$$
$$t_{15} \to -t_{15}$$

all products $[tttt]$ are invariant, so $\mathcal{H}$ is invariant.

We can generalize this to local symmetry under $U(1)$ and under non-Abelian Lie groups G. For $U(1)$, associate with each link a phase



let $\quad V_p = [e^{i\phi_{12}} e^{-i\phi_{32}} e^{-i\phi_{43}} e^{i\phi_{41}}]$



then $\quad \mathcal{H} = \frac{1}{2g^2} \sum_{\text{plaquettes}} [V_p + V_p^\dagger]$

For the non-Abelian case, associate with each link of the lattice a unitary matrix U in a representation of G

$$V = U_{12} U_{32}^\dagger U_{43}^\dagger U_{41}$$

we can set
$$H = \frac{1}{4g^2} \sum_{\text{plaq.}} \mathrm{tr}[V_p + V_p^\dagger]$$

This is invariant under local symmetries

$$U_{ij} \to W_i U_{ij} \qquad U_{ki} \to U_{ki} W_i^\dagger$$

with W a unitary matrix of G. The partition function is

$$Z = \int \underbrace{\mathcal{D}U_{ij}} \; \exp\Big[\frac{1}{4g^2} \sum_{\text{plaq.}} \mathrm{tr}[V_p + V_p^\dagger] \Big]$$

integral over the Lie group G, normalized by $\int \mathcal{D}U_{ij} = 1$

Let's work out the expansion of Z as $g \to 0$, $\frac{1}{g^2} \to \infty$.
In this limit $\mathrm{tr}[V_p] \to 1$, so $U_{ij} \to 1$ or to field configurations equivalent to this by local symmetry

transformations. Write

$$U_{ij} = e^{igA^a_{i\mu}t^a} \qquad \mu = \text{direction } i\to j$$

where $A_{i\mu}$ is small and, as we will see, a smooth function of x_i. Now U is the comparator, familiar from our discussion of gauge symmetry, and $A_\mu(x)$ is the connection. In that discussion, we computed

$$V_p = U\,U^\dagger U^\dagger U = 1 + ig\,F^a_{\mu\nu}t^a + \cdots \qquad \mu\nu = \text{orientation of plaquette}$$

where $F^a_{\mu\nu}$ is the Yang-Mills field strength, and

$$\mathrm{tr}\,V_p = (\text{const}) - \frac{g^2}{2}(F^a_{\mu\nu})^2 + \cdots$$

then

$$e^{-H} \cong e^{-\frac{1}{4}\sum_{i\mu\nu}(F^a_{\mu\nu})^2_i}$$

and the weak-coupling ($g\to 0$) expansion is that of continuum Yang-Mills theory. The theory we have constructed is a lattice-regularized version of Yang Mills theory with exact local gauge invariance — "lattice gauge theory". This theory was discovered by Wilson and Polyakov., the Z_2 version by Wegner.

We can use mean-field theory to look for a phase transition as a function of g. The self-consistency condition is:

$$\langle U_{ab}\rangle = \frac{\int \mathcal{D}U\; e^{\frac{d(d-1)}{24g^2}\mathrm{tr}[(\langle U\rangle)^3(U+U^\dagger)]}\; U_{ab}}{\int \mathcal{D}U\; e^{\frac{d(d-1)}{24g^2}\mathrm{tr}[(\langle U\rangle)^3(U+U^\dagger)]}}$$

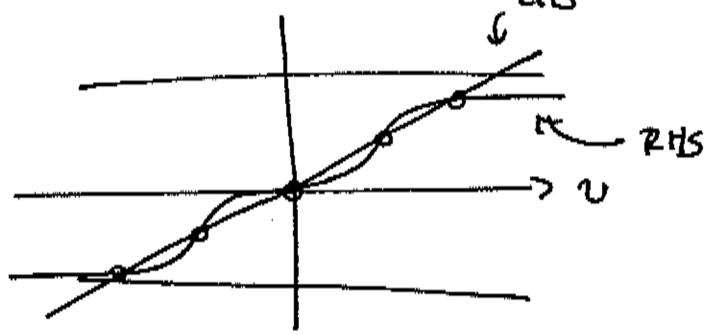
assuming $\langle U_{ab} \rangle = v \cdot \mathbb{1}_{ab}$. This formula gives only the solution $v=0$ for $\frac{1}{g^2} \to 0$, but it has nontrivial solution as $\frac{1}{g^2} \to \infty$. To expand for small $\frac{1}{g^2}$, we need the formulae (for SU(N))

$$\int \mathcal{D}U \; 1 = 1 \qquad \int \mathcal{D}U \; U_{ab} = 0 \qquad \int \mathcal{D}U \; U_{ab} U^{\dagger}_{cd} = \frac{\delta_{ad}\delta_{bc}}{N}$$

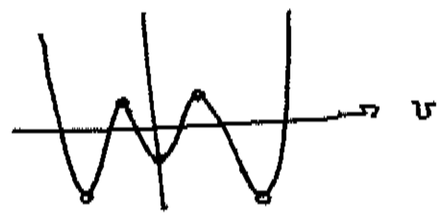
[check on the last expression:

$$\int \mathcal{D}U \; U_{ab} U^{\dagger}_{bd} = \delta_{ad} = \delta_{ad}\frac{\delta_{bb}}{N} \quad \checkmark \quad]$$

You can see that the RHS is cubic in v as $v \to 0$, thus for $\frac{1}{g^2}$ large:



corresponding to a free energy function

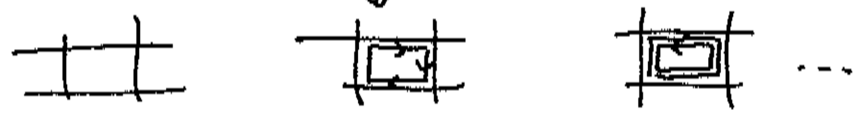


Mean-field theory predicts a first-order phase transition between a low-g phase — where the usual perturbative Yang Mills theory is a qualitative description * and a high-g phase, where the physics is different.

We can understand the "strong coupling", $g\to\infty$, phase by working out an expansion analogous to that on pp. 5-7. Expand Z using

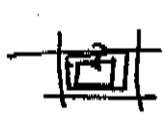
$$e^{\frac{1}{4g^2}\sum_P tr[V_P+V_P^\dagger]} = \prod_P\left(1+\frac{1}{4g^2}tr[V_P+V_P^\dagger]+\cdots\right)$$

diagram this as



analogous to the free energy formula at the bottom of p. 6, we have

$$Z = \exp\Big[N \Big\{ \frac{d(d-1)}{(4g^2)^2} + \frac{d(d-1)(d-2)}{3!}\cdot 2\left(\frac{1}{4g^2}\right)^6 + \cdots$$

 +  + ...

where I have used

$$\int DU \; U_{ab}\,U^\dagger_{cd} = \frac{\delta_{ad}\,\delta_{bc}}{N}$$

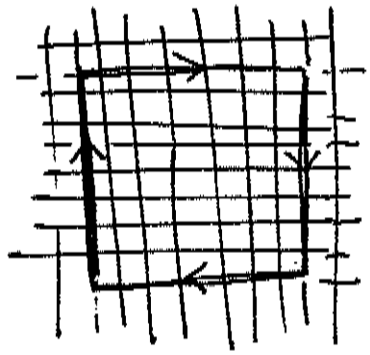
to evaluate integrals over the U's.

The most illuminating computation we did in our earlier discussion was the calculation of the spin-spin correlation function $\langle \vec{S}_0\cdot\vec{S}_I\rangle$. Let me now write an analogue of this object for the lattice gauge theory. We must choose a gauge-invariant function of the U_{ij}. The only such object that is large is the Wilson loop:

$$\langle tr[P\, e^{i\int dx\, g A^a_\mu t^a}]\rangle = \langle W_P[A]\rangle$$

which in the lattice gauge thy becomes:

$$W_P[U] = \prod_P U_{ij}$$



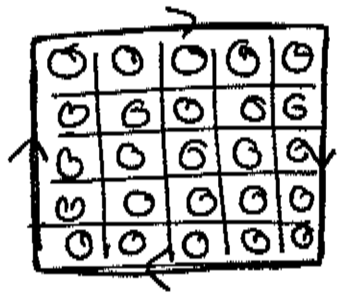
the product of U matrices tracing the path P :

Let's evaluate $\langle W_P[U]\rangle$ at strong coupling.

Since

$$\int DU \; U_{ab} = 0$$

we must bring down a factor of $\frac{1}{4g^2}\,\mathrm{tr}\,[V_p + V_p^\dagger]$ so that every link has at least two factors $U, U^\dagger$ under the integral. This requires that we tile the entire surface spanning P:



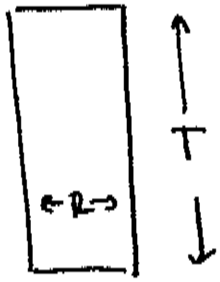
$$\langle W_P \rangle = \prod_{\substack{\text{plaquettes}\\ \text{spanning } P}} \left(\frac{1}{4g^2}\right) = \exp\left[- A \cdot \log(4g^2)\right]$$

Area of loop in plaquettes.

This "Wilson area law" is the characteristic behavior of the strong-coupling region.

In Euclidean field theory, the Wilson loop represents the gauge interaction of heavy sources with Yang-Mills charge.

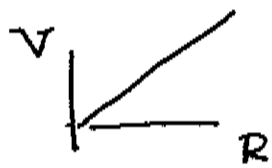
We saw that, for a Wilson loop of dimensions



$$\langle W_P \rangle = \exp\left[- T\, V(R) \right]$$

where $V(R)$ is the potential energy of heavy sources.

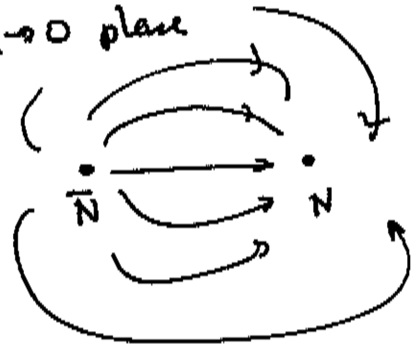
The area law implies $V(R) = (\text{const}) \cdot R$.



so, in the strong coupling region, colored sources are permanently confined into color-singlet bound states!

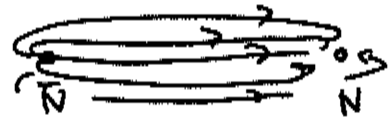
A way of picturing this physics is that the Yang-Mills electric flux that runs from the $\bar{N}$ to the N runs, in the strong-coupling phase, only in the vicinity of the tiling surface. Then:

$g \to 0$ phase



Coulomb potential $\sim \frac{1}{R}$

$g \to \infty$ phase



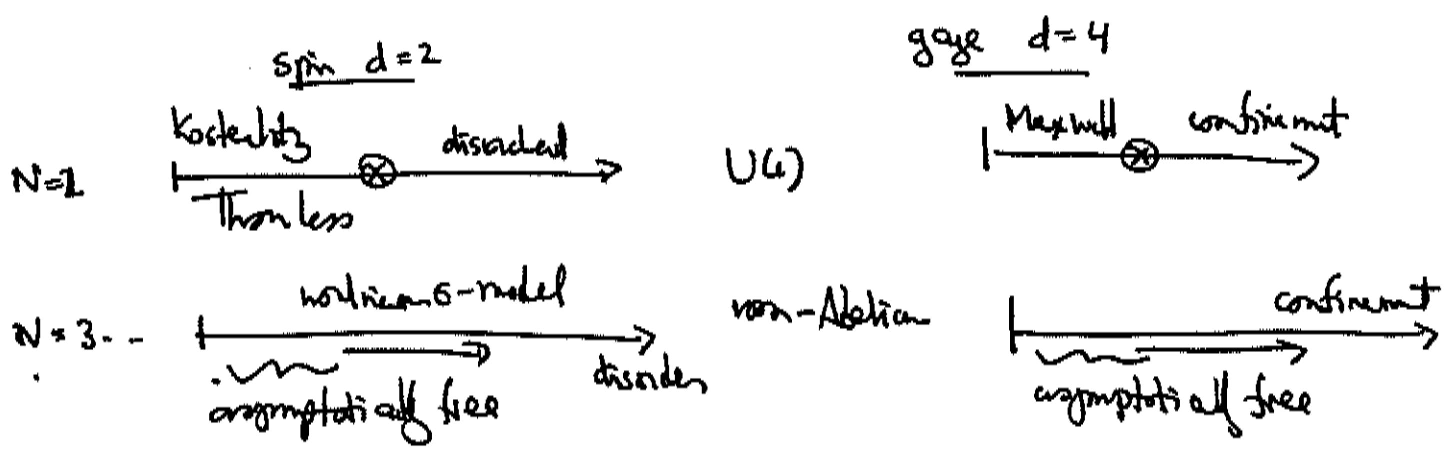
Energy $\sim R$

$V(R) \sim R$

Arguments have been given that the strong-coupling phase is a condensate of Yang-Mills magnetic charge. Then, just as magnetic flux in a superconductor forms narrow flux tubes, Yang-Mills electric flux would form tubes in this phase.

In spin systems we saw that the mean field prediction of a phase transition was not correct in sufficiently low dimensions. In $d=2$, lattice gauge theory can be solved exactly, and it has *no* phase transition. Similarly, in $d=3$ (for $G = U(1)$ or another continuous group) it can be shown that the strong-coupling phase extends all the way down to $g^2=0$. In $d=4$, the $U(1)$ lattice gauge theory has a weak-coupling phase with *photons*, and a phase transition (actually, 2^{nd}-order) at a finite coupling. However, numerical studies show that, for *non-Abelian* gauge groups, there is *no* phase transition. These theories are asymptotically *free*, and it is reasonable that theories with small g^2 flow to $g^2 \to \infty$ at large distances.

There is a nice analogy to spin systems in $d=2$:



This picture coincides nicely with the behavior we need to give a correct qualitative description of QCD: SU(3) sym theory in 4-D, with weak coupling at short distances, should show permanent confinement of color sources at large distances.

For further reading, see:

K.G. Wilson, Phys. Rev. <u>10</u>, 2445 (1974)

M.E. Peskin, Annals of Physics <u>113</u>, 122 (1978)