Instantons

In the previous lecture we derived the Adler-Bell-Jackiw anomaly equations for QED with zero electron mass.

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

and for QCD with n_f massless flavors:

$$\partial_\mu j^{\mu 5} = -\frac{g^2 n_f}{32\pi^2}\varepsilon^{\mu\nu\lambda\sigma}F^a_{\mu\nu}F^a_{\lambda\sigma} \qquad \text{using } \mathrm{tr}\, t^a t^b = \tfrac{1}{2}\delta^{ab}\cdot n_f$$

Do these equations imply that chiral charge is globally not conserved?

$$\Delta(Q_R - Q_L) = \int d^4x \; \partial_\mu j^{\mu 5} \neq 0 \qquad ?$$

In QED, we can argue in the following way. Notice that the right-hand side of the anomaly equation is a total divergence:

$$\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} = 4\varepsilon^{\mu\nu\lambda\sigma}\partial_\mu A_\nu \partial_\lambda A_\sigma = \partial_\mu\left[4\varepsilon^{\mu\nu\lambda\sigma}A_\nu\partial_\lambda A_\sigma\right]$$

assuming that $A_\mu \to 0$ on the boundary at ∞, this term integrates to zero, and the number of massless L and R fermions is separately conserved.

In QCD, the right-hand side of the anomaly equation is also a total divergence:

$$\varepsilon^{\mu\nu\lambda\sigma}F^a_{\mu\nu}F^a_{\lambda\sigma} = 4\varepsilon^{\mu\nu\lambda\sigma}\Big[\partial_\mu A^a_\nu \partial_\lambda A^a_\sigma + g f^{abc}\partial_\mu A^a_\nu A^b_\lambda A^c_\sigma$$
$$+ \frac{1}{4}g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\lambda A^e_\sigma\Big]$$

The last term is zero by the Jacobi identity:

$$f^{ab[c}f^{ade]} = 0 \qquad \text{antisym in } [cde]$$

then

$$\varepsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma} = \partial_\mu \left[4\varepsilon^{\mu\nu\lambda\sigma} \left(A^a_\nu \partial_\lambda A^a_\sigma + \frac{1}{3} f^{abc} A^a_\nu A^b_\lambda A^c_\sigma \right)\right]$$

It is convenient to introduce:

$$\underset{\sim}{A}_\mu = g A^a_\mu t^a \qquad \mathrm{tr}\, t^a t^b = \tfrac{1}{2}\delta^{ab}$$

then

$$= \partial_\mu \left[\frac{8\varepsilon^{\mu\nu\lambda\sigma}}{g^2} \mathrm{tr}\left[\underset{\sim}{A}_\nu \partial_\lambda \underset{\sim}{A}_\sigma - \frac{2i}{3} \underset{\sim}{A}_\nu \underset{\sim}{A}_\lambda \underset{\sim}{A}_\sigma \right]\right]$$

so

$$\partial_\mu j^{\mu 5} = -n_f \cdot \partial_\mu \left[\frac{1}{4\pi^2} \varepsilon^{\mu\nu\lambda\sigma} \mathrm{tr}\left(\underset{\sim}{A}_\nu \partial_\lambda \underset{\sim}{A}_\sigma - \frac{2i}{3} \underset{\sim}{A}_\nu \underset{\sim}{A}_\lambda \underset{\sim}{A}_\sigma \right)\right]$$

If we consider A as a 1-form:

$$A = i\,dx^\mu \underset{\sim}{A}_\mu$$

The object on the right-hand side

$$\Omega_{cs} = \mathrm{tr}\left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right]$$

is called the Chern-Simons form.

Now, isn't it still true that we can integrate the RHS out to the boundary at ∞ and obtain zero? Surprisingly, the answer to this question is no. There are specific field configurations for which we get a specific finite answer. The situation is most clean if we work in 4-d Euclidean space. Then the boundary at ∞ is a large sphere with the topology of S^3, the unit sphere in 4-dimensions. The Euclidean functional integral for Yang Mills theory is:

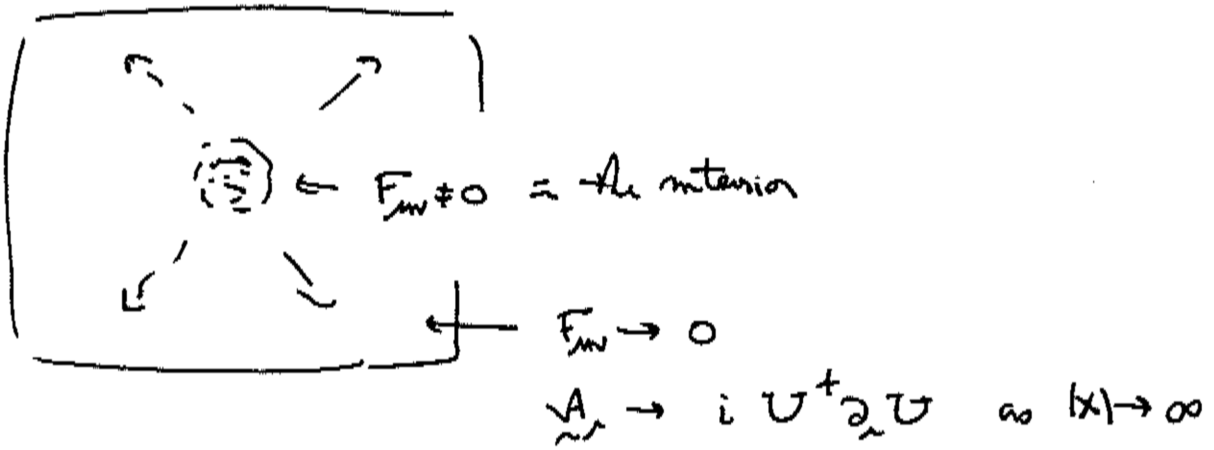
$$\int \mathcal{D}A \; \exp\left[-\int d^4x \; \frac{1}{4}(F^a_{\mu\nu})^2 \right]$$

and this gives zero weight to field configurations in which $F^a_{\mu\nu} \neq 0$ at ∞. But, we can have $F_{\mu\nu} = 0$ with $A_\mu \neq 0$, if A_μ is a pure gauge:

$$\underset{\sim}{A}_\mu = g A^a_\mu t^a = i\, U^\dagger \partial_\mu U$$

where U is a unitary matrix of the gauge group. From here on, I will work with the simplest case, $G = SU(2)$. Since every non-Abelian Lie group has $SU(2)$ as a subgroup, the solutions we find will be present in every Yang-Mills theory with a non Abelian gauge group.

We now seek Euclidean gauge field configurations with



In principle, U can be a function of the position on the sphere at ∞.

Put
$$\underset{\sim}{A}_\mu = i\, U^\dagger \partial_\mu U$$

into the expression for the Chern-Simons form:

$$\partial_{[\lambda} \underset{\sim}{A}_{\sigma]} = i\, \partial_{[\lambda} U^\dagger \partial_{\sigma]} U = -i\, U^\dagger (\partial_{[\lambda} U) U^\dagger (\partial_{\sigma]} U)$$

then:

$$\varepsilon^{\mu\nu\lambda\sigma}\, tr\left[\underset{\sim}{A}_\nu \partial_\lambda \underset{\sim}{A}_\sigma - \frac{2i}{3} \underset{\sim}{A}_\nu \underset{\sim}{A}_\lambda \underset{\sim}{A}_\sigma\right]$$

$$= \left[i\cdot(-i) - \frac{2i}{3}(i)^3\right] \varepsilon^{\mu\nu\lambda\sigma} tr\left[U^+(\partial_\nu U)\, U^+ \partial_\lambda U\, U^+ \partial_\sigma U\right]$$

$$= \frac{1}{3}\, \varepsilon^{\mu\nu\lambda\sigma}\, tr\left[(U^+\partial_\nu U)(U^+\partial_\lambda U)(U^+\partial_\sigma U)\right]$$

then

$$\int d^4x\, \partial_\mu j^{\mu 5} = (-n_f)\cdot \underbrace{\int d^3x\, n_\mu}_{\text{integral over the sphere at } \infty} \frac{1}{12\pi^2}\, \varepsilon^{\mu\nu\lambda\sigma}\, tr\left[(U^+\partial_\nu U)(U^+\partial_\lambda U)(U^+\partial_\sigma U)\right]$$

The expression on the RHS is invariant to global symmetry transformation

$$U(x) \to g\, U(x) \qquad g = \text{global unitary transformation}$$

So we can evaluate it near any fixed element of G and then transform to get the general case. Work in G = SU(2), evaluate the RHS near U = 1 and near the north pole of the sphere at ∞. Actually, SU(2) is isomorphic to the sphere S^3, with the isomorphism expressed by the parametrization

$$U = n^0 + i\,\vec{n}\cdot\vec{\sigma} \qquad (n^0)^2 + |\vec{n}|^2 = 1$$

Near U = 1 (set $U(x) = 1$ at $x = (R, 0, 0, 0)$ on the large sphere:

$$U^+\partial_i U \equiv i\,(\partial_i n^a)\,\sigma^a$$

$$tr\, \sigma^a\sigma^b\sigma^c = 2i\,\epsilon^{abc}$$

$$\text{RHS} \equiv (-n_f)\cdot \int d^3x\, \frac{1}{12\pi^2}\, \epsilon^{ijk}\, 2i\,(i)^3\, \partial_i n^a\, \partial_j n^b\, \partial_k n^c\, \epsilon^{abc}$$

so

$$\text{RHS} = -n_f \left(\frac{2 \cdot 3!}{12\pi^2}\right) \cdot \underbrace{\int d^3A}_{\text{area on the unit sphere} = SU(2) \text{ covered by } U(x).}$$

Now we have an interesting situation. The value of RHS depends on the topology of the mapping $U(x)$ from the unit sphere at ∞ into $SU(2)$. If this mapping is trivial

$$U(x) = \text{constant} \quad \text{or} \quad U(x) = \text{contractible to a constant}$$

the net area covered is zero. However, if

$$U(x) = \text{corresponding point on } S^3$$

— or a mapping topologically equivalent to this — we obtain

$$\int d^3A = \text{area of } S^3 = 2\pi^2$$

In general, let the Pontryagin number

$$n = \#\text{ of times that } U(x) \text{ covers } SU(2) \cong S^3.$$

then

$$\int d^4x\, \partial_\mu j^{\mu 5} = -n_f \cdot (2n)$$

It is convenient to denote

$$\tilde{F}^{\mu\nu a} = \tfrac{1}{2} \varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}^{a}$$

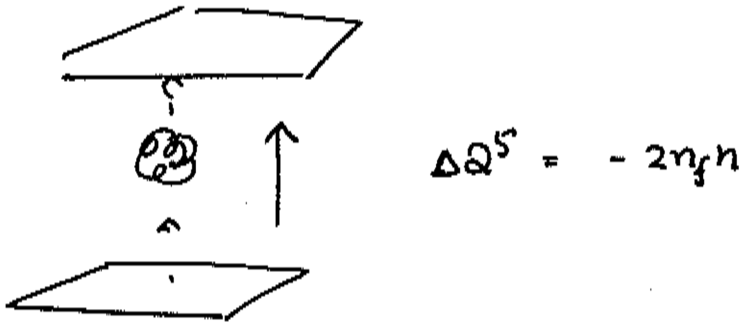
then

$$\partial_\mu j^{\mu 5} = -\frac{g^2 n_f}{16\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$$

The argument we have just given shows that, for *finite-action* configurations in 4-d Euclidean space

$$\int d^4x \; F^a_{\mu\nu}\tilde{F}^{\mu\nu a} = \frac{32\pi^2}{g^2}\cdot n$$

We can think of these field configurations as *processes* in which the gauge fields go through some motions and affect the state of the quark fields



Such a field configuration is called an *instanton*.

More precisely, an *instanton* is the field configuration of minimum action with the required nontrivial topology. This will be a *topologically stable* solution of the Euclidean field equations. (In this sense, the instanton is similar to the *magnetic monopole* solution that we discussed earlier in the course.) We can expand the Euclidean functional integral about this solution to work out its quantum effects.

Let's now construct the instanton solution explicitly. We seek the minimum action Euclidean field configuration with

$$\int d^4x \, (F^a_{\mu\nu} \tilde{F}^{\mu\nu a}) = \frac{32\pi^2}{g^2}$$

To find it, we can use the Bogomolny trick: Note that, in Euclidean space

$$\tilde{F}^a_{\mu\nu} \tilde{F}^{\mu\nu a} = \frac{1}{4} \varepsilon_{\mu\nu\lambda\sigma} \varepsilon^{\mu\nu\alpha\beta} F^a_{\lambda\sigma} F^{\alpha\beta a} = (F^a_{\lambda\sigma})^2$$

$(\varepsilon^{0123} = \varepsilon_{0123} = +1$ in Euclidean space)

then

$$0 \le \int d^4x \, (F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu})^2 = \int d^4x \; 2 \cdot ((F^a_{\mu\nu})^2 \pm F^a_{\mu\nu} \tilde{F}^{\mu\nu a})$$

So the Euclidean action

$$\frac{1}{4} \int d^4x \, (F^a_{\mu\nu})^2 \ge \pm \frac{1}{4} \int d^4x \; F^a_{\mu\nu} \tilde{F}^{\mu\nu a}$$

and

$$\frac{1}{4} \int d^4x \, (F_{\mu\nu})^2 = \frac{1}{4} \int d^4x \; F^a_{\mu\nu} \tilde{F}^{\mu\nu a}$$

when

$$F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu} \qquad \text{"self-dual field configuration"}$$

This is a <u>first</u>-order differential equation for A^a_μ.

The solution to this equation was found by Belavin, Polyakov, Schwarz, and Tyupkin. To write the solution simply, the following symbol is helpful:

$$\eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \mu,\nu = 1,2,3 \\ -\delta_{a\mu} & \mu = 1,2,3 \quad \nu = 0 \\ +\delta_{a\nu} & \nu = 1,2,3 \quad \mu = 0 \\ 0 & \mu = \nu = 0 \end{cases}$$

In terms of group theory a is an index of the adjoint rep. of $SU(2)$, $[\mu\nu]$ is an index of the adjoint rep of $SO(4)$. Now

$$SO(4) \cong SU(2)\times SU(2)$$

and $\eta_{a\mu\nu}$ is the projector onto the first $SU(2)$.

$$\bar{\eta}_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \mu,\nu = i,j = 1,2,3 \\ +\delta_{a\mu} & \mu = i \quad \nu = 0 \\ -\delta_{a\nu} & \nu = i \quad \mu = 0 \\ 0 & \mu = \nu = 0 \end{cases}$$

projects onto the other $SU(2)$. You can check these identities for the η's:

$$\tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}\,\eta_{a\lambda\sigma} = \eta_{a\mu\nu} \qquad \text{self-dual}$$

$$\tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}\,\bar{\eta}_{a\lambda\sigma} = -\bar{\eta}_{a\lambda\sigma} \qquad \text{anti-self-dual}$$

$$\eta_{a\mu\nu}\,\eta_{a\lambda\sigma} = \delta_{\mu\lambda}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\lambda} + \epsilon_{\mu\nu\lambda\sigma}$$

$$\eta_{a\mu\nu}\,\eta_{b\mu\nu} = 4\delta_{ab}$$

$$\epsilon^{abc}\,\eta_{b\mu\nu}\,\eta_{c\lambda\sigma} = \delta_{\mu\lambda}\,\eta_{a\nu\sigma} - \delta_{\mu\sigma}\,\eta_{a\nu\lambda} - \delta_{\nu\lambda}\,\eta_{a\mu\sigma} + \delta_{\nu\sigma}\,\eta_{a\mu\lambda}$$

Now let me propose:

$$A^a_\mu = \frac{2}{g}\,\frac{\eta_{a\mu\nu}x^\nu}{x^2+\rho^2} \qquad \text{or} \qquad \underset{\sim}{A}_\mu = \frac{2\eta_{a\mu\nu}x^\nu\,\frac{\sigma^a}{2}}{x^2+\rho^2}$$

Compute $F^a_{\mu\nu}$:

$$\partial_\mu A^a_\nu = \frac{2}{g}\left[\frac{\eta_{a\nu\mu}}{x^2+\rho^2} - \frac{2x^\mu\, \eta_{a\nu\lambda}\, x^\lambda}{(x^2+\rho^2)^2}\right]$$

$$-\partial_\nu A^a_\mu = \frac{2}{g}\left[-\frac{\eta_{a\mu\nu}}{x^2+\rho^2} + \frac{2x^\nu\, \eta_{a\mu\lambda}\, x^\lambda}{(x^2+\rho^2)^2}\right]$$

$$g\,\epsilon^{abc} A^b_\mu A^c_\nu = \frac{4}{g}\,\epsilon^{abc}\,\eta_{b\mu\lambda}\,\eta_{c\nu\sigma}\,\frac{x^\lambda x^\sigma}{(x^2+\rho^2)^2}$$

$$= \frac{2}{g}\frac{1}{(x^2+\rho^2)^2}\Big[2\,\delta_{\mu\nu}\,\underbrace{\eta_{a\lambda\sigma}\,x^\lambda x^\sigma}_{=0} - 2\,\delta_{\mu\sigma}\,\eta_{a\lambda\nu}\,x^\lambda x^\sigma$$
$$- 2\,\delta_{\lambda\nu}\,\eta_{a\mu\sigma}\,x^\lambda x^\sigma + 2\,\delta_{\lambda\sigma}\,\eta_{a\mu\nu}\,x^\lambda x^\sigma\Big]$$

~ all

$$F^a_{\mu\nu} = \frac{2}{g}\frac{1}{(x^2+\rho^2)}\Big\{-2\,\eta_{a\mu\nu}(x^2+\rho^2) - 2\eta_{a\nu\lambda}\,x^\mu x^\lambda$$
$$+ 2\,\eta_{a\mu\lambda}\,x^\nu x^\lambda$$
$$+ 2\,\eta_{a\nu\lambda}\,x^\mu x^\lambda - 2\,\eta_{a\mu\sigma}\,x^\nu x^\sigma$$
$$+ 2x^2\,\eta_{a\mu\nu}\Big\}$$

$$F^a_{\mu\nu} = \frac{-4\rho^2\,\eta_{a\mu\nu}}{g\,(x^2+\rho^2)^2}$$

since $\eta_{a\mu\nu}$ is self-dual, this is self dual and solves the condition on p. 7.

We can write this solution more generally as

$$A^a_\mu = R_{ab} \frac{2}{g} \frac{\eta_{a\mu\nu}(x-x_0)^\nu}{(x-x_0)^2+\rho^2}$$

where R_{ab} is a rotation, which can equally well be considered to act in the SU(2) gauge group or as a rotation of the 4-d space. In all, this solution has 8 parameters:

x_0	center	4
ρ	size	1
R_{ab}	orientation	3
		8

It is worth checking:

$$\int d^4x \, (F_{\mu\nu})^2 = \int d^4x \, (F\tilde{F})$$

$$= \int d^4x \quad \frac{16\rho^4}{g^2} \frac{\eta_{a\mu\nu}\eta_{a\mu\nu}}{(x^2+\rho^2)^4}$$

$$= \int_0^\infty \frac{dx^2 \, x^2 \cdot \pi^2}{(x^2+\rho^2)^4} \quad \frac{16 \cdot 12}{g^2} \rho^4$$

$$= \frac{16 \cdot 12\pi^2}{g^2} \int_0^\infty dx^2 \left(\frac{\rho^4}{(x^2+\rho^2)^3} - \frac{\rho^6}{[x^2+\rho^2]^4} \right)$$

$$= \frac{16 \cdot 12\pi^2}{g^2} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{32\pi^2}{g^2} \quad \text{as } \underline{\text{predicted}}!$$

How do we use these solutions in the evaluation of the functional integral?

$$Z = \int \mathcal{D}A \; e^{-\int d^4x \; tr(F)^2}$$

the integral is actually a sum of integrals over the different topological sectors

$$= \underbrace{\int \mathcal{D}A}_{n=0} \; e^{-\int d^4x \; tr(F_n)^2} + \underbrace{\int \mathcal{D}A}_{n=1} \; e^{-\int d^4x \; tr(F_n)^2} + \cdots$$

the sector $n=0$ gives the usual Yang-Mills perturbation theory. In the sector $n \neq 0$, the minimum value of the action is

$$\int d^4x \; \frac{1}{4} (F^a_{\mu\nu})^2 \Big|_{min} = \frac{8\pi^2}{g^2} \cdot |n|$$

so these sectors give non-perturbative corrections. In particular, the contribution in the sector $n=1$ is an expansion about the instanton solution

$$\underbrace{\int \mathcal{D}A}_{n=1} \; e^{-\int d^4x \; tr(F_{\mu\nu})^2} \cong \int d^4x_0 \, \frac{d\rho}{\rho^5} \int d^3\Omega \;\; e^{-8\pi^2/g^2} \cdot (\det[\;])^{-\frac{1}{2}}$$

where $(\det[\;])^{-\frac{1}{2}}$ is the result of integrations over the fluctuations about the instanton solution. One effect of this is to replace

$$\frac{8\pi^2}{g^2} \rightarrow \frac{8\pi^2}{g^2(\rho)}$$

where $g(\rho)$ is the running QCD coupling at the scale ρ. The factor $\frac{1}{\rho^5}$ is required by dimensional analysis.

Continue the evaluation of Z, which gives the vacuum energy:

$$Z = \exp\Big[- \underset{\substack{\uparrow \\ \text{4-volume}}}{(\text{Vol})_4} \cdot \underset{\substack{\uparrow \\ \text{vacuum energy density}}}{\mathcal{E}_0} \Big]$$

The $n=-1$ sector gives a similar contribution from anti-instantons. For the $n=+2$ and -2 sectors we can write

$$\underbrace{\int \mathcal{D}A}_{n=2}\, e^{-\int d^4x \frac{1}{4}(F^a_{\mu\nu})^2} \approx \frac{1}{2!}\Big[\int d^4x_0 \frac{d\rho}{\rho^5} \int d\Omega\, e^{-8\pi^2/g^2(\rho)} \cdot C \Big]^2$$

from two widely-separated instantons, plus a smaller correction when the instantons come close together. There is also a non-perturbative correction to the $n=0$ sector from widely separated instantons and anti-instantons. Keeping all of the contributions in which instantons and anti-instantons are widely separated, we find

$$Z = \sum_{n_+,n_-} \frac{1}{n_+!\,n_-!}\Big[\int d^4x_0 \frac{d\rho}{\rho^5} d\Omega\, e^{-8\pi^2/g^2(\rho)} \cdot C \Big]^{n_+ + n_-}$$

$$= \exp\Big[-(\text{Vol})_4 \cdot (-2) \cdot \Big(\int \frac{d\rho}{\rho^5} d\Omega\, e^{-8\pi^2/g^2(\rho)} \cdot C \Big) \Big]$$

so instantons give a negative, nonperturbative contribution to the vacuum energy. The integral as I have written it is actually divergent as $\rho \to 0$ $g^2(\rho) \to \infty$. But small instantons, anyway, give a well-defined contribution.

These considerations motivate adding a term to the action of QCD:

$$\mathcal{L} = +\frac{1}{4}(F^a_{\mu\nu})^2 \quad i\,\theta\,\frac{g^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$$

Actually, this term should have been there from the beginning: It is of dimension 4 and is gauge invariant. It is usually omitted because it gives 0 in perturbation theory. However, in a sector w. Pontryagin number n

$$\int d^4x \;\frac{g^2}{32\pi^2} F\tilde{F} = n$$

and this term gives a weight

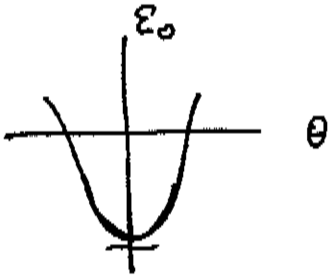
$$e^{-\int \mathcal{L}\, d^4x} = e^{-\frac{8\pi^2}{g^2}|n|}\; e^{in\theta}$$

in the functional integral. With the θ term, the expression for Z on the previous page becomes:

$$Z = \sum_{n_+ n_-} \frac{1}{n_+!\,n_-!}\left[\int d^4x_0 \int \frac{d\rho}{\rho^5}\, d\Omega\; e^{-8\pi^2/g^2(\rho)}\cdot C\right]^{n_+ + n_-} e^{i\theta(n_+ - n_-)}$$

$$= \exp\left[-(\mathrm{Vol})_4\;(-2\cos\theta)\cdot\int \frac{d\rho}{\rho^5}\, d\Omega\; e^{-8\pi^2/g^2(\rho)}\cdot C\right]$$

So the vacuum energy depends on θ:



θ must be a physical parameter of QCD. It is a somewhat troublesome parameter, because it is CP-odd, but that is another story...

I would now like to return to the equation

$$\Delta Q^5 = \int d^4x\, \partial_\mu j^{\mu 5} = -2 n_f \cdot n$$

If we have the explicit instanton solution, we can study how this nonconservation of Q^5 is realized. It must come from a property of the fermionic functional integral

$$\int \mathcal{D}\psi \mathcal{D}\bar\psi \;\; e^{i\int \bar\psi i \not{D} \psi}$$

with the instanton field in the background. Let's study this for $n_f = 1$, one quark flavor.

We are still in Euclidean space. Choose γ matrices

s.t. $$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$$

i.e. $$\gamma^0 = \left(\begin{array}{c|c} & 1 \\ \hline 1 & \end{array}\right) \qquad \gamma^j = \left(\begin{array}{c|c} & -i\sigma^j \\ \hline i\sigma^j & \end{array}\right) \qquad \gamma^\mu = \left(\begin{array}{c|c} & \sigma^\mu \\ \hline \bar\sigma^\mu & \end{array}\right)$$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{array}{c|c} -1 & \\ \hline & 1 \end{array}\right)$$

$$\psi = \left(\begin{array}{c} \psi_L \\ \hline \psi_R \end{array}\right) \qquad \bar\psi = (\bar\psi_R, \bar\psi_L)$$

so

$$\bar\psi i \not{D} \psi = \bar\psi_L\, i\bar\sigma\cdot D\, \psi_L + \bar\psi_R\, i\sigma\cdot D\, \psi_R$$

$i\bar\sigma\cdot D$, $i\sigma\cdot D$ are not self-adjoint, but they are adjoints of one another!

$$\int d^4x\ \bar\Psi_L\, i\bar\sigma\cdot D\, \psi_L = \int d^4x\ \bar\Psi_L\ i\,(D_0 + i\bar\sigma\cdot\vec D)\,\psi_L$$

$$= \int d^4x\,(-i)\,[\partial_0\bar\Psi_L + i\,g\,\bar\Psi_L A_0^a t^a$$
$$+ i\,\vec\partial\,\bar\Psi_L\,\vec\sigma - g\,\bar\Psi_L\,\vec A^a\vec\sigma\cdot t^a]\,\psi_L$$

so $$[(i\bar\sigma\cdot D)^\dagger\,\bar\Psi_L]^T = +i\,\{[\partial_0 - ig\,A_0^a(-t_a^T)]\,\bar\Psi_L^T$$
$$+ i(\vec\sigma)^T\cdot[\vec\partial - ig\vec A^a(-t_a^T)]\ \bar\Psi_L^T\}$$

multiply by σ^2 $\qquad \sigma^2\ \vec\sigma^T = -\vec\sigma\ \sigma^2$

$$\sigma^2\,(i\bar\sigma\cdot D)^\dagger\ \sigma^2 = +i\,[(\partial_0 - ig\,A_0^a\ t^a_{\bar r}) - i\,\vec\sigma\cdot(\vec\partial - ig\vec A^a t^a_{\bar r})]$$

so $\quad (i\bar\sigma\cdot D)^\dagger \cong i\sigma\cdot D \qquad$ with $r\to\bar r$

and similarly

$\quad (i\sigma\cdot D)^\dagger \cong i\bar\sigma\cdot D$

Since for the $\underline{2}$ of SU(2) $\quad 2\cong\bar 2\quad$ we have $(i\bar\sigma\cdot D)^\dagger \cong i\sigma\cdot D$ and vice versa.

Now look at the eigenvectors and eigenvalues of these operators:

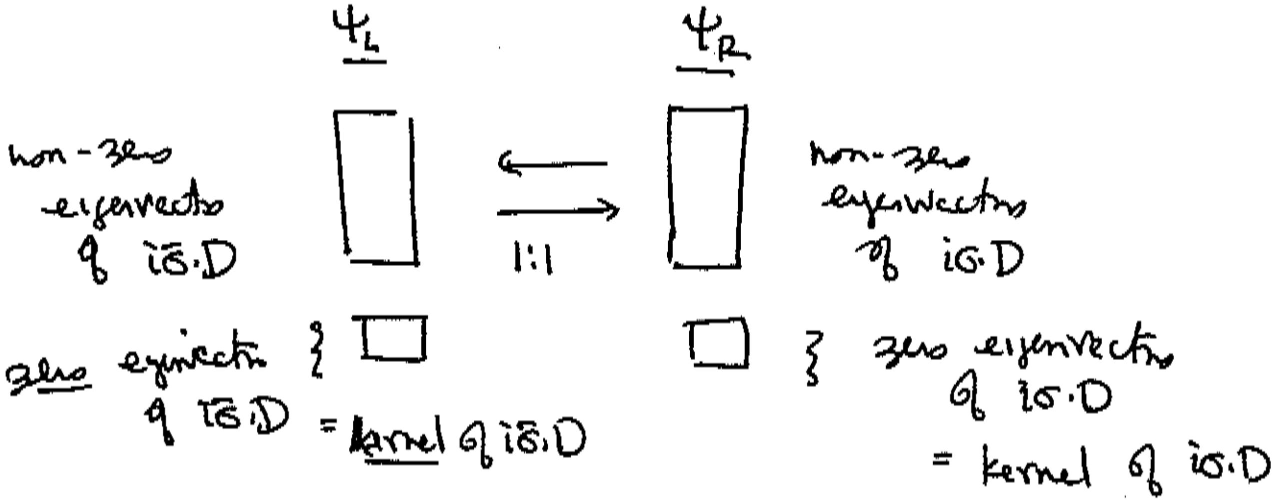
$$(i\bar\sigma\cdot D)\,\psi_{Lk} = \bar\lambda_k\,\psi_{Lk} \qquad \bar\Psi_{L\ell}\ i\bar\sigma\cdot D = \lambda_\ell\,\bar\Psi_{L\ell}$$
$$(i\sigma\cdot D)\,\psi_{Rk} = \lambda_\ell\,\psi_{R\ell} \qquad \bar\Psi_{Rk}\ i\sigma\cdot D = \bar\lambda_k\,\bar\Psi_{Rk}$$

As indicated, the spectra of $i\bar{\sigma}\cdot D$ from the left and $i\sigma\cdot D$ from the right are identical, and vice versa. It is tempting to say that, also, the λ_ℓ and the $\bar{\lambda}_k$ are equal:

$$\int d^4x \; \bar{\psi}_{L\ell}\, (i\bar{\sigma}\cdot D)\, \psi_{Lk}$$

$$= \quad \bar{\lambda}_k \int d^4x \; \bar{\psi}_{L\ell}\, \psi_{Lk} \;=\; \lambda_\ell \int d^4x \; \bar{\psi}_{L\ell}\psi_{Lk}$$

so if $\int d^4x \; \bar{\psi}_{L\ell}\psi_{Lk} \neq 0$ $\quad \lambda_\ell = \bar{\lambda}_k$. Then all <u>non</u>zero λ_ℓ and $\bar{\lambda}_k$ are paired. However, if $\lambda_\ell = 0$, it need not have a partner, and the same for $\bar{\lambda}_k$. We then get the following picture of the spectra of these operators:



We define

$$\text{index}[\,i\bar{\sigma}\cdot D\,] = \text{dimension}\,(\,\ker\,(i\bar{\sigma}\cdot D)) - \text{dimension}\,(\ker(i\sigma\cdot D))$$

$$= (\#\text{ of zero eigenvectors of } i\bar{\sigma}\cdot D) - (\#\text{ of zero eigenvectors of } i\sigma\cdot D)$$

I would now like to show that $i\vec{\sigma}\cdot D$ has a zero eigenvector in the field of an instanton.

We need to find a solution of

$$(D_0 + i\vec{\sigma}\cdot\vec{D})\,\psi_L = \bar{\lambda}\,\psi_L \qquad \text{with} \quad \underline{\bar{\lambda} = 0}$$

Try an expression

$$\psi_{L\alpha i} = \frac{A_{\alpha i}}{[x^2+\rho^2]^{\underline{a}}} \leftarrow \text{constant matrix}$$

$\alpha = 1,2$ Lorentz index

$i = 1,2$ SU(2) color index

$$D_\mu = \partial_\mu - ig A^a_\mu t^a = \partial_\mu - i\frac{2}{(x^2+\rho^2)}\eta_{a\mu\nu}x^\nu\frac{\sigma^a}{2}$$

$$(D_0 + i\vec{\sigma}\cdot D)\,\psi_L = \frac{1}{[x^2+\rho^2]^{\underline{a}+1}}\Big\{ -2ax^0 - i\eta_{a0\nu}x^\nu\sigma^a + i\sigma^j(-2ax^j) + \eta_{aj\nu}x^\nu\sigma^j\sigma^a \Big\}\cdot A$$

Look at the terms with x^0

$$\{\ \}\Big|_{x^0} = x^0(-2\underline{a} + \eta_{aj0}\sigma^j\sigma^a)\,A$$

$$= x^0(-2\underline{a} - \vec{\sigma}\cdot\vec{\sigma})\,A = 0$$

Now $A_{\alpha i}$ is a matrix that combines two spin-½'s. We can have

$A_{\alpha i} \propto \epsilon_{\alpha i}$ ½×½ → spin 0

$A_{\alpha i}$ $\underline{\text{symmetric}}$ ½×½ → spin 1

the total J^2 is

$$J^2 = \Big(\underbrace{\frac{\vec{\sigma}}{2}}_{\text{Lorentz}} + \underbrace{\frac{\vec{\sigma}}{2}}_{SU(2)}\Big)^2 = \frac{3}{4} + \frac{1}{2}\vec{\sigma}\cdot\vec{\sigma} + \frac{3}{4} = \frac{3}{2}\Big(1 + \frac{\vec{\sigma}\cdot\vec{\sigma}}{3}\Big)$$

So

$\underline{\text{spin } 0} \Rightarrow \vec{\sigma}\cdot\vec{\sigma} = -3$ spin 1 $\Rightarrow \vec{\sigma}\cdot\vec{\sigma} = +1$

for the spin 0 case $\quad \underline{a} = 3/2$

For the spin 1 case $\quad \underline{a} = -1/2 \quad \times$ a non-normalizable wavefunction

Note that if we were working with $\psi_R \qquad (D_0 - i\vec{\sigma}\cdot\vec{D})\psi_R = 0$,

we would have found $\quad (-2\underline{a} + \vec{\sigma}\vec{\sigma}) = 0$

$\underline{\text{spin } 0} \quad a = -3/2 \qquad \underline{\text{spin } 1} \quad a = +1/2$

neither of which gives a normalizable wave function. So, we find a zero mode only for ψ_L, not for ψ_R:

To finish the argument, check the x^j term:

$$\{\ \}\big|_{x^k} = (-i\eta_{a0k}\, x^k \sigma^a + i\sigma^j(-2a x^j) + \eta_{ajk}\, x^k \sigma^j \sigma^a)\, A$$

$$= (-i\,\vec{x}\cdot\vec{\sigma} - 2ai\,\vec{x}\vec{\sigma} + \epsilon_{kaj}\, x^k \sigma^j \sigma^a)\, A$$

$$(\sigma^j)_{\alpha\beta}\,(\sigma^a)_{ij}\,\varepsilon_{\beta j} = [(\sigma^j)(\varepsilon)(\sigma^a)^T]_{\alpha i} = [\sigma^j(-\sigma^a)\varepsilon]_{\alpha i}$$

$$= -(\sigma^j\sigma^a)_{\alpha\beta}\,\varepsilon_{\beta i}$$

and $$\epsilon_{kaj}\,(-\sigma^j\sigma^a) = \epsilon_{kaj}(-1)\, i\epsilon^{jal}\sigma^l = 2i\,\delta^{kl}\sigma^l$$

so

above $= (-i\,\vec{x}\cdot\vec{\sigma})(1 - 2a + 2) = 0$ for $a = 3/2$ ✓

so $$\psi_{L\alpha i} = \frac{\varepsilon_{\alpha i}}{[x^2+\rho^2]^{3/2}}$$ is a zero mode of $i\bar{\sigma}\cdot D$

There is no corresponding zero mode of $\psi_{R\alpha i}$, $i\sigma\cdot D$.

So, for the instanton field $\quad \text{index}\,(i\bar{\sigma}\cdot D) = 1$

The index has the important property that it is a topological invariant, in the following sense: The index of $i\bar{\sigma}\cdot D$ is unchanged under small deformations of the A_μ field. It is possible that, as we vary A_μ, a zero mode could become a nonzero mode, or vice versa. However, since nonzero modes are always paired, a zero mode of $i\bar{\sigma}\cdot D$ can only become nonzero if a zero mode of $i\sigma\cdot D$ also goes from zero to nonzero. Similarly for transitions from nonzero to zero modes.

This insight extends to the following remarkable theorem:

Atiyah-Singer index theorem: $\text{index}(i\bar{\sigma}\cdot D) = \frac{g^2}{32\pi^2}\int d^4x\, F\tilde{F} = n$

Pontryagin number

Then, in the sector n of the A field functional integral, all field configurations have ($n>0$) n zero modes of ψ_L; and ($n<0$) $|n|$ zero modes of ψ_R, plus possible paired zero modes.

Let's now evaluate the integral over ψ in the sector $n=1$, taking the zero mode into account. The fermionic action is

$$\int d^4x\, \bar{\psi}\, i\not{D}\psi$$

expand ψ and $\bar{\psi}$ in eigenstates of $i\not{D}$: Let ψ_{Lk}, ψ_{Rk}, $\bar{\psi}_{Lk}, \bar{\psi}_{Rk}$ be paired nonzero eigenstates, $\psi_{L0}, \bar{\psi}_{R0}$ be the zero modes, all of these c-number functions. Let $c_k, \bar{c}_k, d_k, \bar{d}_k$ be corresponding Grassmann parameters: Then:

$$\psi_L(x) = c_0 \psi_{L0}(x) + \sum_k c_k \psi_{Lk}(x)$$

$$\psi_R(x) = \sum_k d_k \psi_{Rk}(x)$$

with

$$\bar\psi_L(x) = \sum_k \bar c_k \bar\psi_{Lk}(x) \qquad \bar\psi_{Lk} = (\sigma^2 \psi_{RL})^T$$

$$\bar\psi_R(x) = \sum_k d_k \bar\psi_R(x) + \bar d_0 \bar\psi_R(x) \qquad \bar\psi_{Rk_0} = (\sigma^2 \psi_{Lk,0})^T$$

Then $\int d^4x \, \bar\psi i \not{D} \psi = \sum_k \lambda_k [\bar c_k c_k + \bar d_k d_k]$ $c_0, \bar d_0$ do not appear

assuming that the wavefunctions are normalized: $\int d^4x \, \bar\psi_{Lk} \psi_{L\ell} = \delta_{k\ell}$ etc.

The functional measure is

$$\int D\psi D\bar\psi = \int dc_0 \prod_k dc_k \, d\bar c_k \; d\bar d_0 \prod_k d\bar d_k \, dd_k$$

So

$$\int D\psi D\bar\psi \, e^{-\int d^4x \, \bar\psi i \not{D} \psi} = \prod_k (\lambda_k^2) \cdot \int dc_0 \, d\bar d_0$$

Our rule for fermionic integration is $\int dc_0 \cdot 1 = 0$, so the contribution of the n=1 sector in the presence of a massless fermion is zero.

Now compute

$$\langle \bar\psi_R \psi_L(x) \rangle = \frac{\int DA\,D\psi \, e^{-\int \mathcal{L}} \, \bar\psi_R \psi_L(x)}{\int DA\,D\psi \, e^{-\int \mathcal{L}}}$$

In perturbation theory this order parameter violates Q^5 conservation, so $\langle \bar\psi_R \psi_L \rangle = 0$. However, in the n=1 sector

$$\int \underbrace{\mathcal{D}A}_{n=1} \mathcal{D}\psi \;\; e^{-\int \mathcal{L}} \;\; \bar\psi_R \psi_L(x)$$

$$= \int \underbrace{\mathcal{D}A}_{n=1} \;\; e^{-\int \frac{1}{4}(F_{\mu\nu})^2} \;\; e^{i\theta} \;\; \int \mathcal{D}\psi \;\; \bar\psi_R(x)\, \psi_L(x)\, e^{-\sum_k \lambda_k(\bar c_k c_k + \bar d_k d_k)}$$

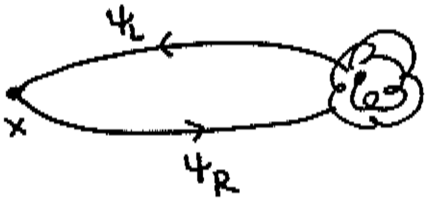
$$= \int \underbrace{\mathcal{D}A}_{n=1} \;\; e^{-\int \frac{1}{4}(F_{\mu\nu})^2} e^{i\theta} \;\; \underbrace{\det(i\not{D})}_{\text{nonzero modes}} \;\; \int dc_0 \, d\bar d_0 \;\; \bar\psi_R(x)\, \psi_L(x)$$

$$= \int \underbrace{\mathcal{D}A}_{n=1} \;\; e^{-\int \frac{1}{4}(F_{\mu\nu})^2} e^{i\theta} \;\; \underbrace{\det(i\not{D})}_{\text{nonzero modes}} \;\; \underbrace{\bar\psi_{R0}(x)\, \psi_{L0}(x)}_{\text{zero modes!}}$$

expand about the instanton field:

$$= \int d^4x_0 \int \frac{d\rho}{\rho^5}\, d\Omega \;\; C \;\; e^{-8\pi^2/g^2(\rho)} \; e^{i\theta} \;\; \frac{(\text{const})}{((x-x_0)^2+\rho^2)^3}$$

The instanton is effectively an operator $\underline{\bar\psi_L}\psi_R$ at x_0



that changes Q^5 by $\underline{\Delta Q^5 = -2}$! Anti-instantons give the h.c. of this operator.

This mechanism – discovered by 't Hooft – explicitly realizes the equation

$$\Delta Q^5 = -2 n_f n$$

For n_f flavors, the instanton creates one ψ_L and destroys one ψ_R of each flavor. This corresponds to the effective operator:

$$\det(\bar\psi_{Li}\psi_{Rj}) \qquad i,j = 1 — n_f$$

The classical theory with n_f massless flavors has the global symmetry

$$\underbrace{U(n_f)}_{\psi_{Li}} \times \underbrace{U(n_f)}_{\psi_{Ri}}$$

The instanton effective interaction breaks this to

$$\underbrace{U(1)}_{\text{baryon no}} \times \underbrace{SU(n_f)}_{\text{vector}} \times \underbrace{SU(n_f)}_{\text{axial}}$$

i.e. it breaks only $U_A(1)$ or Q^5. When the axial symmetries are spontaneously broken by quark mass generation, we get

$$n_f^2 - 1 \qquad \underline{\text{not}} \qquad n_f^2$$

Goldstone bosons, as required for a correct strong interaction phenomenology.

The nonperturbative breaking of axial U(1) has many consequences. Here is an unusual one: In the Standard Model, the baryon number current has an SU(2) anomaly. Since the weak interaction SU(2) couples only to left-handed quarks:

$$\partial_\mu \Big(\sum_i \bar q \gamma^\mu q \Big) = \underset{\uparrow\!\_\_\_\_}{+} \frac{g^2}{32\pi^2} \cdot \underbrace{\frac{1}{2}}_{\substack{\uparrow \\ \text{from } L \ \left(\frac{1-\gamma^5}{2}\right) \\ \text{not } R}} \cdot n_f \cdot F^a_{\mu\nu} \tilde F^{\mu\nu a}$$

With n_g = # of generations = $2n_f$, an instanton process in the weak-interaction SU(2) gives:

$\Delta B = + n_g \cdot n$ $\quad$ n = Pontryagin no.

$\Delta L = + n_g \cdot n$ $\quad$ n_g = # of generations = 3

Under normal conditions, the rate of these processes is very small

$$e^{-\frac{8\pi^2}{g^2}} = e^{-2\pi/\alpha_w} \sim e^{-180}$$

However, at high temperature, the transition corresponding to $n \neq 0$ can be thermally excited. We need temperatures well above the electroweak phase transition: $T \gg \text{TeV}$.

However, at these high temperatures, processes with

$$\Delta B = \Delta L \neq 0$$

occur. Such processes could be important in determining the baryon density produced in the early universe.

for further reading:

G. 't Hooft Phys. Rev. Lett. 37, 8 (1976)
Phys. Rev. D14, 3432 (1976)
E D16 2199 (1978)