Solitons and Magnetic Monopoles in Unified Gauge Theories

So far in this course, we have discussed only the properties of spontaneously broken gauge theories only at the level of perturbation theory. However, these theories often contain additional structure that is essentially nonperturbative. In this lecture, I will introduce this structure with some examples in 1 and 3 dimensions. [A beautiful reference: E.B. Bogomolny, Sov. J. Nucl Phys. 24, 449 (1976).]

Begin with the simple model of ϕ^4 theory – with a real scalar field, in 1+1 dimensions

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \left(-\tfrac{1}{2}\mu^2\phi^2 + \tfrac{\lambda}{4}\phi^4\right)$$

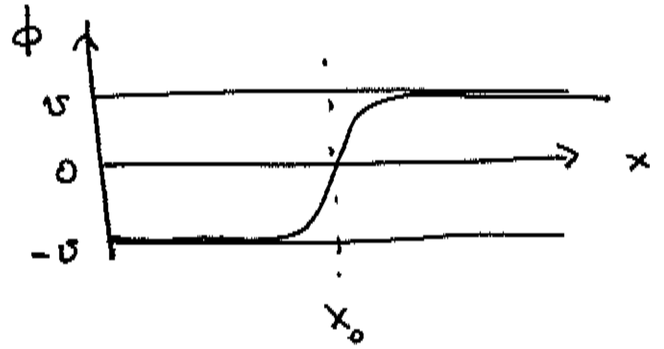
The potential of this theory has the form: 

The vacuum states are

$$\langle \phi \rangle = \pm v = \pm\sqrt{\mu^2/\lambda}$$

There are two degenerate vacuum states. Note that $v \sim 1/\sqrt{\lambda}$, so $v \gg |\mu|$ and the vacua are well separated for small λ.

Now consider a classical field configuration of the form:



This is a domain wall: For $x < x_0$, ϕ is in the vacuum $\langle\phi\rangle = -v$. For $x > x_0$, ϕ is in the vacuum $\langle\phi\rangle = +v$.

It is not difficult to find the classical field configuration explicitly. The equation of motion for ϕ is

$$-\partial^2\phi + \mu^2\phi - \lambda\phi^3 = 0$$

Let $\phi = v f(z)$ where $z = m_\phi(x-x_0) = \sqrt{2}\mu(x-x_0)$ (time-independent). Then

$$2\frac{d^2}{dz^2}f + f - f^3 = 0$$

The function

$$f(z) = \tanh z/2$$

satisfies this equation with the correct boundary conditions $\phi(x) \to v$ as $x \to \infty$ $\to -v$ as $x \to -\infty$. More explicitly

$$\phi = v \tanh\left[\frac{m_\phi(x-x_0)}{2}\right]$$

there is an arbitrary center x_0, Away from x_0, the deviation from the vacuum falls off exponentially

$$x > x_0 \qquad \phi \sim v - O(e^{-m_\phi(x-x_0)})$$

like a QFT correlation function. The energy of the configuration is

$$E = \int dx \left[\tfrac{1}{2}(\partial_x\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \tfrac{\lambda}{4}\phi^4 \right]$$

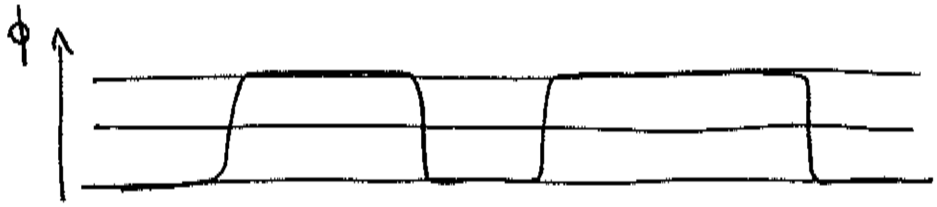
$$= \int dx \left(\frac{\mu^2 v^2}{4}\right)\Big[\frac{2}{\cosh^4 m_\phi x/2} \underbrace{- 1}_{\text{energy of the broken-symmetry vacuum}} \Big]$$

$$= \underbrace{\frac{2\sqrt{2}}{3}\mu v^2}_{} - E_0$$

the domain wall has a localized energy $\propto \frac{1}{\lambda}$. This is very large as $\lambda \to 0$. Such a massive, localized field configuration in a nonlinear field theory is called a soliton.

This state should be interpreted as a localized massive particle with mass given by the energy above. The perturbative corrections to the energy are small, down by one power of λ. So the result is unambiguous.

There are states in the field theory that contain many of these particles:



The massive particles carry a Z_2 quantum number; a pair of these particles can annihilate to purely perturbative excitations. An isolated particle, however, is absolutely stable. We say that the classical field configuration is "topologically stable": It cannot decay to small fluctuations about a perturbative vacuum consistent with the topology of its boundary conditions.

I will now show that similar topologically stable massive particles can be found in spontaneously broken gauge theories. The construction is due to 't Hooft and Polyakov.

Consider the Georgi - Glashow model: an SO(3) gauge theory spontaneously broken by a Higgs field in the vector representation of SO(3).

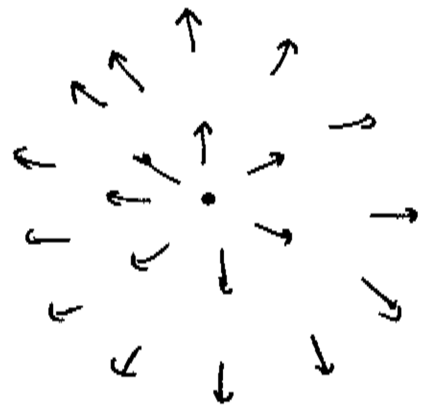
$$\langle \phi^a \rangle = v (\hat{n})^a \qquad a = 1,2,3$$

Two of the three vector bosons of SO(3) receive mass

$$m_W = g v$$

The third vector boson $A_\mu^a \hat{n}^a$ remains massless; this is the photon of a unified theory of weak and electromagnetic interactions.

Now consider an alternative Higgs field configuration of non-trivial topology:



$\phi^a(x) = v f(r) \hat{r}^a$

This is a non-singular configuration if $f(r) \to 0$ as $r \to 0$.

The boundary condition for the gauge field at large distances can be found by analyzing the Higgs kinetic energy term: $\frac{1}{2} (D_\mu \phi)^2$.

$$D_\mu \phi^a = \partial_\mu \phi^a + g \epsilon^{abc} A^b_\mu \phi^c$$

$$\partial_i \hat{r}^a = \partial_i \frac{r^a}{(r^2)^{1/2}} = \frac{\delta^{ia} - \hat{r}^i \hat{r}^a}{r} = \frac{\epsilon^{abc} (\epsilon^{ibk} \hat{r}^k) \hat{r}^c}{r}$$

If $D_\mu \phi^a \sim \frac{1}{r}$, we obtain $\int d^3x \; \frac{1}{2} (D\phi)^2 \sim \int d^3x \frac{1}{r^2} = \infty$.

So the $g A_\mu \phi$ term must cancel this leading $\frac{1}{r}$ term. We accomplish this if

$$A^b_i = - \frac{\epsilon^{ibk} \hat{r}^k}{g r} h(r) \quad , \; h(r) \to 1 \text{ as } r \to \infty .$$

For a non-singular solution $h(r) \sim r^2$ as $r \to 0$

This ansatz

$$\phi^a(r) = v f(r) \hat{r}^a \qquad A_i^b = -\frac{\epsilon^{ibk}}{gr}\hat{r}^k h(r)$$

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w. $f(r) \to 1$, $h(r) \to 1$ as $r \to \infty$, leads to a topologically stable solution.

It is interesting to compute the Yang-Mills field tensor for this configuration:

$$F_{0i}{}^a = 0 \qquad \text{since} \qquad \frac{d}{dt} = 0 \qquad A_0 = 0$$

but $F_{ij}{}^a$ is nonzero:

$$F_{ij}{}^a = \partial_i A_j^a - \partial_j A_i^a + g\,\epsilon^{abc} A_i^b A_j^c$$

$$= \partial_i \left[-\frac{\epsilon^{jal}}{gr}\hat{r}^l h \right] - (i \leftrightarrow j) + \frac{\epsilon^{abc}}{gr^2}\epsilon^{ibl}\hat{r}^l \epsilon^{jcm}\hat{r}^m h^2$$

$$= \left(-\frac{\epsilon^{jal}}{g}\,\frac{\delta^{il} - 2\hat{r}^i\hat{r}^l}{r^2} h - \frac{\hat{r}^i \epsilon^{jal}\hat{r}^l}{gr} h' \right) - (i \leftrightarrow j)$$

$$+ \frac{1}{gr^2}\left(\delta^{ia}\delta^{cl} - \delta^{ic}\delta^{al}\right)\epsilon^{jcm}\hat{r}^l\hat{r}^m h^2$$

$$= \left(-\frac{(\epsilon^{ija} - 2\hat{r}^i\epsilon^{jal}\hat{r}^l)}{gr^2} h - \frac{\hat{r}^i\epsilon^{jal}\hat{r}^l}{gr} h' \right) - (i \leftrightarrow j)$$

$$+ \frac{1}{gr^2}\epsilon^{ijm}\hat{r}^a\hat{r}^m h^2$$

This is a little hard to parse. It will be easier if we construct

$$B^{ka} = -\tfrac{1}{2}\epsilon^{ijk} F_{ij}{}^{a} \qquad \text{the YM magnetic field}$$

$$B^{ka} = \left[\frac{\delta^{ka}}{gr^2} h - (\delta^{ak}\delta^{li} - \delta^{ai}\delta^{lk}) \frac{h\hat{r}^i\hat{r}^l}{gr^2} + \tfrac{1}{2}(\delta^{ak}\delta^{li} - \delta^{ai}\delta^{lk}) \frac{\hat{r}^i\hat{r}^l h'}{gr} \right]$$

$$\times 2$$

$$- \frac{1}{gr^2}\delta^{km}\hat{r}^a\hat{r}^m h^2$$

$$= \cancel{\frac{2}{gr^2}\delta^{ka} h} - \cancel{\frac{2}{gr^2}\delta^{ka} h} + \frac{2\hat{r}^a\hat{r}^k}{gr^2} h$$

$$+ \frac{\delta^{ak} - \hat{r}^a\hat{r}^k}{gr} h' - \frac{\hat{r}^a\hat{r}^k}{gr^2} h^2$$

so

$$B^{ka} = \hat{r}^a \frac{\hat{r}^k}{gr^2}(2h - h^2) + \left(\frac{\delta^{ak} - \hat{r}^a\hat{r}^k}{gr}\right) h'$$

Assuming that $h \to 1$ very rapidly and $h' \to 0$ as $r \to \infty$

$$B^{ka} \to \hat{r}^a \frac{\hat{r}^k}{gr^2}$$

The physical electromagnetic $\vec{B}$ field is the field along the direction of ϕ^a:

$$B^k = \frac{\phi^a}{v} B^{ka} \to \frac{\hat{r}^k}{gr^2}$$

This is a radial $1/r^2$ field: a magnetic monopole!
An electric monopole in this theory has

$$E^k = \frac{r^k}{4\pi r^2} q_E \qquad \text{or} \qquad \int_{\text{sphere at } \infty} d\hat{n}^2 \cdot \vec{E} = q_E$$

For a unit charge $q_E = g$. The soliton solution has

$$\int_{\text{sphere at } \infty} d\hat{n}^2 \cdot B = q_M \qquad \text{where} \qquad q_M = \frac{4\pi}{g}$$

Note that the Y-M-Maxwell equations are satisfied everywhere; in particular

$$\vec{D} \cdot \vec{B}^a = 0$$

But because ϕ^a changes direction on the sphere at ∞,

$$\vec{B} = \frac{\Phi^a}{v} \vec{B}^a$$

can have a nonzero integral.

It is possible to solve explicitly for $f(r)$ and $h(r)$ in a certain limit. The energy of the field configuration is

$$E = \int d^3x \left[\tfrac{1}{2}(B^{ka})^2 + \tfrac{1}{2}(D_i \phi^a)^2 + V(\phi^2) \right]$$

assuming that the configuration is time-independent: $\frac{d}{dt} = 0$, $\vec{E}^a = 0$.
We can write V as:

$$V(\phi^a) = \frac{\lambda}{4} \left[(\Phi^a)^2 - v^2 \right]^2 + \text{const.}$$

Now I would like to consider the case $\lambda \to 0$ (the "Bogomolny - Prasad - Sommerfield (BPS) limit.").

In this case, we can rewrite E as

$$E = \int d^3x \left\{ \frac{1}{2} (D_i\phi^a - B_i^a)^2 + (D_i\phi^a) B_i^a \right\}$$

Since $\partial_i (\phi^a B_i^a) = (D_i\phi^a) B_i^a + \phi^a \underbrace{(D_i B_i^a)}_{=0}$ by $\epsilon^{\mu\nu\lambda\sigma} D_\nu F_{\lambda\sigma} = 0$

the second term is a total divergence.

So if

$$D_i\phi^a = B_i^a \qquad (*)$$

then
$$E = \int d^3x \ \partial_i (\phi^a B_i^a) = \int d^2n^i \ \phi^a B_i^a$$

$$= v \cdot g_m$$

$(*)$ is a set of 1^{st}-order differential equations for f and h !

$$D_i\phi^a = \partial_i (v\hat{r}^a f(r)) + g\epsilon^{abc} \left(-\frac{\epsilon^{ibk}}{gr} \hat{r}^k h\right)(v\hat{r}^c f)$$

$$= v f \frac{\delta^{ia} - \hat{r}^i\hat{r}^a}{r} + v\underline{\hat{r}^a\hat{r}^i} f'$$

$$- \left(\frac{\delta^{ia} - \hat{r}^i\hat{r}^a}{r}\right) v h f$$

$$= \hat{r}^a\hat{r}^i \cdot v f' + \frac{\delta^{ia} - \hat{r}^i\hat{r}^a}{r} v (1-h) f(r)$$

Equating this to p. 7 we find

$$v f' = \frac{2h - h^2}{g r^2} \qquad \frac{h'}{g r} = \frac{v(1-h) f}{r}$$

We can scale out gv by writing $z = g v r = m_W r$

$$\frac{d}{dz} f = \frac{h(2-h)}{z^2} \qquad \frac{d}{dz} h = (1-h) f$$

The solution to these equations is

$$f(z) = \frac{\cosh z}{\sinh z} - \frac{1}{z} \qquad h(z) = 1 - \frac{z}{\sinh z}$$

Notice that $h'(z) \sim 2z e^{-z} \longrightarrow 0$ as $r \to \infty$ as promised. In fact, all of the nontrivial parts of the solution die off exponentially outside of the core of the monopole. The decay is

$$e^{-m_W r}$$

similar to that for the domain wall. Outside the core, we have a pure magnetic monopole.

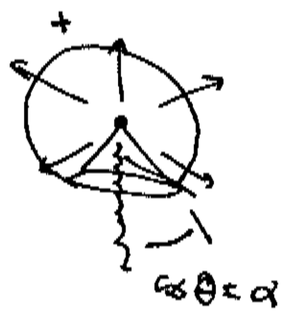
The mass of the monopole is

$$E = v\, g_m = \frac{4\pi v}{g} = \frac{4\pi}{g^2} m_W$$

This mass is $\frac{1}{\alpha} \times$ mass of small oscillations, again, as we found for the domain wall.

It is worth discussing a bit the quantization of the magnetic charge. Naively, it seems that, in a pure Abelian gauge theory with $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{\nabla} \cdot \vec{B} = 0$ strictly. However, Dirac pointed out that it is possible to have properly quantized magnetic charges. The trick is to treat space the way you treat a curved manifold: Cover it with coordinate patches in which $\vec{A}$ is nonsingular; relate $\vec{A}$ in these patches by gauge transformations in the overlap region. In particular, for a magnetic monopole of strength g_M:

$$\vec{B} = \frac{\hat{r}}{4\pi r^2} \cdot g_M$$



The gauge field

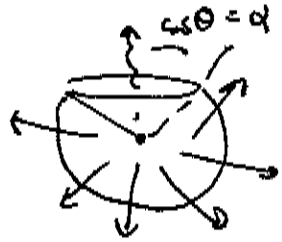
$$\vec{A}_+ = \frac{g_M}{4\pi} \frac{1 - \cos\theta}{r \sin\theta} \hat{\phi}$$

yields $\vec{\nabla} \times \vec{A}_+ = \vec{B}$ and is nonsingular for $\cos\theta > -\alpha > -1$,

and

$$\vec{A}_- = - \frac{g_M}{4\pi} \frac{1 + \cos\theta}{r \sin\theta} \hat{\phi}$$

yields $\vec{\nabla} \times \vec{A}_- = \vec{B}$ and is nonsingular for $1 > \alpha > \cos\theta$



In the overlap region

$$-\alpha < \cos\theta < \alpha$$

both $\vec{A}_+$ and $\vec{A}_-$ are well-defined and

$$\vec{A}_+ - \vec{A}_- = \frac{g_M}{4\pi} \frac{2}{r \sin\theta} \hat{\phi} = \vec{\nabla} \left(\frac{g_M}{2\pi} \phi \right)$$

Locally, this is a gauge transformation. However, globally in the overlap region, this gauge transformation might not be well defined. Let $\psi(x)$ be a field charged under the Abelian gauge theory with electric charge q_E. Then a gauge transformation transforms

$$\psi(x) \to e^{i q_E \cdot \beta(x)} \psi(x) \qquad \vec{A} \to \vec{A} + \vec{\nabla}\beta(x)$$

So we can transform away the difference between $\vec{A}_+$ and $\vec{A}_-$ if, for every such $\psi(x)$, the transformation

$$\psi(x) \to e^{i \frac{q_E q_M}{2\pi} \Phi} \psi(x)$$

is well-defined. This requires

$$\frac{q_E q_M}{2\pi} = n \qquad \text{an } \underline{\text{integer}}$$

for every possible value of q_E in the theory. Dirac said, if there is a magnetic monopole, electric charge is quantized; but we could also start with a theory with quantized electric charge — such as the Georgi-Glashow model — and derive the quantization of magnetic charge. We find in this case

$$q_M = 2\pi n \frac{1}{q_{E0}}$$

where q_{E0} is the <u>minimum</u> electric charge in the theory.

For the Georgi-Glashow model, this is found by coupling a field in the spinor (2) representation of SU(2) to the vector fields; that gives

$$q_{E0} = \frac{g}{2} \qquad q_M = 2\pi n \cdot \frac{2}{g}$$

Indeed, the 't Hooft-Polyakov monopole has $q_M = \frac{4\pi}{g}$!

The 't Hooft-Polyakov solution can be generalized to a soliton with both magnetic and electric charge (the "Julia-Zee dyon"). To do this, consider a time-independent field configuration with

$$\phi^a(x) \qquad A_i^a(x) \qquad A_0^a(x) \qquad \text{nonzero.}$$

Despite the fact that we have no time-dependence,

$$D_0\phi^a = \partial_0\phi^a + g\,\epsilon^{abc}A_0^b\phi^c = g\,\epsilon^{abc}A_0^b\phi^c$$

$$\begin{aligned} E^{ia} &= -\partial_i A_0^a - \partial_0 A_i^a - g\,\epsilon^{abc}A_i^b A_0^c \\ &= -D_i A_0^a \end{aligned}$$

may be nonzero. The energy of the configuration is

$$E = \int d^3x \left\{ \tfrac{1}{2}(E^{ia})^2 + \tfrac{1}{2}(B^{ia})^2 + \tfrac{1}{2}(D_0\phi^a)^2 + \tfrac{1}{2}(D_i\phi^a)^2 + V(\phi) \right\}$$

Let's go to the BPS limit $\lambda \to 0$ or $V(\phi) \to 0$.

Then we can rewrite E as

$$E = \int d^3x \left\{ \tfrac{1}{2}(D_0\phi^a)^2 + \tfrac{1}{2}\left(E^{ia} - C\, D_i\phi^a\right)^2 + \tfrac{1}{2}\left(B^{ia} - (1-C^2)^{1/2} D_i\phi^a\right)^2 \right.$$

$$\left. + \; C\, E^{ia} D_i\phi^a \quad + (1-C^2)^{1/2} B^{ia} D_i \phi^a \right\}$$

for an as-yet-arbitrary constant C. The last two terms are pure surface terms:

$$\int d^3x \; (1-C^2)^{1/2} B^{ia} D_i\phi^a = \int d^2\vec{n} \cdot \vec{B}^a \phi^a (1-C^2)^{1/2} - \int d^3x\, (1-C^2)^{1/2} \underbrace{(D_i B^{ia})}_{0} \phi^a$$

$$= (1-C^2)^{1/2} \; v \cdot q_M$$

$$\int d^3x \;\; C\, E^{ia} D_i \phi^a = \int d^2\vec{n} \cdot \vec{E}^a \phi^a \cdot C - \int d^3x\, C \cdot (D_i E^{ia}) \phi^a$$

$$= C\, v\, q_E$$

since $\phi^a D_i E^{ia} = \phi^a \rho^a \sim \phi^a \epsilon^{abc} \phi^b D_0 \phi^c = 0$. Then we can minimize E subject to the boundary condition of fixed electric and magnetic charges by solving

$$D_0 \phi^a = 0$$

$$E^{ia} = C\, D_i \phi^a$$

$$B^{ia} = (1-C^2)^{1/2} D_i\phi^a$$

Now $D_0\phi^a = g\, \epsilon^{abc} A_0^b \phi^c$ so this $= 0$ if $A_0^b = a\phi^b$

$$E^{ia} = -D_i A_0^a = -a\, D_i\phi^a$$

so the second equation is solved if $a = -C$.

Finally, the third equation can be solved from the field configuration of a pure monopole

$$\phi_0^a(\vec{x}) \qquad A_{i0}^a(\vec{x})$$

by writing

$$\phi^a(x) = \phi_0^a\left((1-c^2)^{1/2} \vec{x} \right)$$

$$A_i^a(x) = (1-c^2)^{1/2} A_{i0}^a\left((1-c^2)^{1/2} \vec{x} \right)$$

then

$$D_i\phi^a = (1-c^2)^{1/2} (D_i\phi^a)_0 \left((1-c^2)^{1/2} \vec{x} \right)$$

$$B^{ia} = \left[(1-c^2)^{1/2} \right]^2 (B^{ia})_0 \left((1-c^2)^{1/2} \vec{x} \right)$$

and we may use $(D_i\phi^a)_0 = (B^{ia})_0$!

The B field tends as $r \to \infty$ to

$$B^{ia} \sim (1-c^2) \frac{\hat{r}^i \hat{r}^a}{g r^2 (1-c^2)} \sim \frac{\hat{r}^i \hat{r}^a}{g r^2} = \frac{g_M}{4\pi r^2} \hat{r}^i \hat{r}^a$$

as before, and

$$E^{ia} \sim \frac{c}{(1-c^2)^{1/2}} \frac{\hat{r}^i \hat{r}^a}{g r^2} \sim \frac{g_E}{4\pi r^2} \hat{r}^i \hat{r}^a$$

so we must identify

$$\frac{c}{(1-c^2)^{1/2}} = \frac{g_E}{g_M} \qquad \text{or} \qquad c = \frac{g_E}{\sqrt{g_E^2 + g_M^2}}$$

for consistency. From this we can determine:

$$E = v \cdot \left(C q_E + (1-C^2)^{1/2} q_M \right)$$

or

$$E = v \left[q_E^2 + q_M^2 \right]^{1/2}$$

It can be shown (this is a somewhat advanced analysis) that q_E is quantized in units of g when the soliton is considered in its quantum field theory; then the dyon of charge m has mass

$$M = v \left[(mg)^2 + \left(\frac{4\pi}{g^2} \right)^2 \right]^{1/2}$$

There are more surprises further along this road. It turns out that the BPS limit of this model can be embedded into an $N=2$ supersymmetric gauge theory. In that theory, the formula

$$E = v \left[q_E^2 + q_M^2 \right]^{1/2} = v \left[(mg)^2 + \left(\frac{4\pi n}{g} \right)^2 \right]^{1/2}$$

is an exact consequence of supersymmetry and is true to all orders of perturbation theory and beyond. Other solitons in supersymmetric theories also have this "BPS" property. For more details, see J. Harvey, hep-th/9603086 (1996).