Grand Unification

Are you disappointed that the Yang-Mills gauge group of the Standard Model is $SU(3) \times SU(2) \times U(1)$? It seems that there are too many components, both in the gauge group and in the fermion matter representations. Can we make it simpler?

Here is a proposal: The group $SU(3) \times SU(2) \times U(1)$ is, in quite natural way, a subgroup of $SU(5)$. Maybe we can write a theory of spontaneously broken $SU(5)$ Yang-Mills theory that leads to the Standard Model. In 1973, Georgi and Glashow showed how to do this, and Pati and Salam constructed a very similar theory based on the group $SO(10)$ that naturally has $SU(5)$ as a subgroup.

To begin, assume that we have an $SU(5)$ Y-M theory with a Higgs field in the adjoint representation of $SU(5)$. This Higgs $\underset{\sim}{\Phi}$ is a traceless Hermitian matrix. A possible potential for Φ is

$$V(\Phi) = -\mu^2 \operatorname{tr} \Phi^2 + \lambda_1 (\operatorname{tr} \Phi^2)^2 + \lambda_2 \operatorname{tr} \Phi^4$$

The symmetric state $\langle \Phi \rangle = 0$ is unstable. Φ can always be diagonalized: $\langle \Phi \rangle = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & a_4 & \\ & & & & a_5 \end{pmatrix}$

with $\sum_i a_i = 0$. Adding this constraint as a Lagrange multiplier, the extrema of V are given by

$$\frac{\partial}{\partial \Phi}\left(-\mu^2 \operatorname{tr}\Phi^2 + \lambda_1 (\operatorname{tr}\Phi^2)^2 + \lambda_2 \operatorname{tr}\Phi^4 + \eta \operatorname{tr}\Phi \right) = 0$$

or

$$\eta + [-2\mu^2 + 4\lambda_1 (\operatorname{tr}\Phi^2)]a_i + 4\lambda_2 a_i^3 = 0$$

So the a_i take at most three different values. Actually, the minima of V are configurations in which the a_i take two values:

$$\langle\Phi\rangle = A \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \quad \text{or} \quad \langle\Phi\rangle = B \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -4 \end{pmatrix}$$

depending on the values of λ_1 and λ_2. The first choice gives a $\langle\Phi\rangle$ that commutes with the matrices

$$\begin{pmatrix} t^a & \\ & \end{pmatrix} \qquad t^a \quad 3\times 3 \text{ Hermitian}$$

$$\begin{pmatrix} & \\ & \tau^a \end{pmatrix} \qquad \tau^a \quad 2\times 2 \text{ Hermitian}$$

$$c \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

The covariant derivative on Φ is:

$$D_\mu \Phi = \partial_\mu \Phi - i g_5 A_\mu^A [t^A, \Phi]$$

so the generators of SU(5) corresponding to these matrices are left unbroken and the associated A_μ^A are left massless.

The remaining generators of SU(5) $\left(\begin{array}{c|c} & X \\ \hline X^+ & \end{array}\right)$

are spontaneously broken, and the associated gauge bosons receive mass $\quad m_X = 5 g_5 A$. We'll come back to these bosons later.

The unbroken symmetry generators generate the group $SU(3)\times SU(2)\times U(1)$. To define g_5 consistently, all generators should be normalized to a common convention

$$\mathrm{tr}[t^A t^B] = \tfrac{1}{2}\,\delta^{AB}$$

Then t^a, τ^a will be the standard SU(3) and SU(2) matrices, and the last matrix will be

$$\sqrt{\tfrac{3}{5}}\left(\begin{array}{ccc|cc} -\tfrac{1}{3} & & & & \\ & -\tfrac{1}{3} & & & \\ & & -\tfrac{1}{3} & & \\ \hline & & & +\tfrac{1}{2} & \\ & & & & +\tfrac{1}{2} \end{array}\right) \equiv \sqrt{\tfrac{3}{5}}\; Y$$

In a moment, I will identify Y with the hypercharge of the GSW weak interaction theory. So, the part of the SU(5) covariant derivative with massless gauge fields is

$$D_\mu = \partial_\mu - i g_5 \underbrace{t^a A^a_\mu}_{SU(3)} - i g_5 \underbrace{A^a_\mu \tau^a}_{SU(2)} - i \underbrace{\sqrt{\tfrac{3}{5}} g_5}\, B_\mu Y$$

so we will predict an $SU(3)\times SU(2)\times U(1)$ gauge theory with

$$g_s = g \qquad g' = \sqrt{\tfrac{3}{5}}\, g$$

or $\tan\theta_w = \dfrac{g'}{g} = \sqrt{\tfrac{3}{5}}$ $\quad \sin^2\theta_w = \tfrac{3}{8}$. Both predictions are wrong for the Standard Model, but wait a minute; we'll fix it.

Now couple fermions to this model. The simplest choice is a multiplet of left-handed fermions in the $\underset{\sim}{5}$ of SU(5). This is acted on the unbroken generators as

$$\left(\begin{array}{c|c} t^a & \\ \hline & \end{array}\right)\left(\begin{array}{c} \\ \hline \\ \end{array}\right)\begin{array}{l} \}\ 3 \text{ of } SU(3) \\ \}\ 1\text{'s of } SU(3) \end{array}$$

$$\left(\begin{array}{c|c} & \\ \hline & \tau^a \end{array}\right)\left(\begin{array}{c} \\ \hline \\ \end{array}\right)\begin{array}{l} 1\text{'s of } SU(3) \\ 2 \text{ of } SU(2) \end{array}$$

$$\left(\begin{array}{ccc|cc} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ \hline & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{array}\right)\left(\begin{array}{c} \\ \hline \\ \end{array}\right)\begin{array}{l} \}\ Y = -\frac{1}{3} \\ \}\ Y = \frac{1}{2} \end{array}$$

These do not correspond to any Standard Model fermions. However, the $\bar{5}$ of SU(5) gives:

$$\left(\begin{array}{c} \\ \hline \\ \end{array}\right)\begin{array}{l} \}\ \bar{3} \text{ of } SU(3),\ 1 \text{ of } SU(2),\ Y = +\frac{1}{3} \\ \}\ 1 \text{ of } SU(3)\ \ 2 \text{ of } SU(2)\ \ Y = -\frac{1}{2} \end{array}$$

These are the $(\bar{d})_L$ = antiparticle of d_R

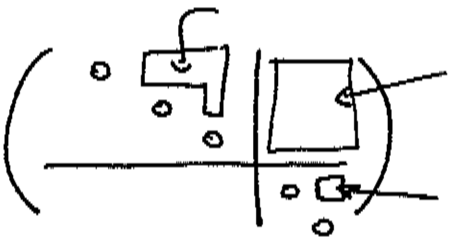
$$L_L = \begin{pmatrix} \nu \\ e \end{pmatrix}$$

The other fermions of the Standard Model can be found in the

5 × 5 antisymmetric tensor representation of SU(5)

($\frac{5\cdot 4}{2}$ = 10-dimensional)

1 of SU(2) $(3\times 3)_{\text{antisym}} = \bar{3}$ of SU(3) $Y = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$



3 of SU(3), 2 of SU(2)
$Y = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$

1 of SU(3)
$(2\times 2)_{\text{antisym}} =$ 1 of SU(2)
$Y = \frac{1}{2} + \frac{1}{2} = 1$

These are the $(\bar{u})_L$, $Q = \begin{pmatrix} u \\ d \end{pmatrix}_L$, $(e^+)_L$. So, left-handed fermions in the $\bar{5} + 10$ representation of SU(5) account for 1 full generation of fermions in the Standard Model. Since $\bar{5} + 10$ is not a real representation of SU(5), this model implies parity violation — but (by arguments in Peskin & Schroeder section 20.3) only in the weak interaction.

Notice that, because SU(5) is a simple group, the U(1) that results from spontaneous symmetry breaking is quantized. After SU(2) × U(1) breaking, this will give

$$Q(\text{proton}) = -Q(e^-) \quad \text{exactly!}$$

The SU(5) structure also gives all of the bizarre Y assignments for the various quarks and leptons.

Return now to the question of the coupling constants. What can we do about this? If the grand unified group is broken at a very high mass scale, there might be substantial running of coupling constants between there and accessible particle physics scales. The effect goes in the right direction: the SU(3) coupling should get stronger, the U(1) coupling should get weaker. To be more quantitative write

$$\alpha_i = \frac{g_i^2}{4\pi} \qquad\qquad \begin{aligned} g_1 &= \sqrt{\tfrac{5}{3}}\, g' \\ g_2 &= g \\ g_3 &= g_s \end{aligned}$$

$$\frac{1}{\alpha_i}(Q) = \frac{1}{\alpha_i}(M) + \frac{b_i}{2\pi} \log\left(\frac{Q}{M}\right)$$

we computed

$$b = \frac{11}{3} C_2(G) - \sum_{\substack{\text{left-hand}\\\text{fermions}}} \frac{2}{3} C(r_f) - \sum_{\substack{\text{complex}\\\text{bosons}}} \frac{1}{3} C(r_b)$$

Let m_U be the scale of grand unified ("GUT") symmetry breaking. For $Q \ll m_U$, only the particles that do not obtain masses $\sim m_U$ are relevant. A first hypothesis would be that these are only the particles of the Standard model. Then we can evaluate the b_i: In the following,

n_g is the number of fermion generations:

SU(3): $$b_3 = \underbrace{\frac{11}{3}\cdot 3}_{\text{gluons}} - \frac{2}{3}\cdot 2\cdot \overset{C(r)}{\frac{1}{2}}\cdot \underbrace{2}_{u+d}\cdot n_g$$

SU(2): $$b_2 = \frac{11}{3}\cdot \underbrace{2}_{SU(2)\text{ bosons}} - \frac{2}{3}\cdot\frac{1}{2}\cdot\underbrace{(3+1)}_{Q+L}\cdot n_g$$

U(1) $$b_1 = 0 - \frac{2}{3}\cdot\frac{3}{5}\Big(2\cdot 3\cdot\left(\tfrac{1}{6}\right)^2 + 3\left(\tfrac{1}{3}\right)^2 + 3\left(-\tfrac{2}{3}\right)^2 + \underbrace{2\cdot\left(-\tfrac{1}{2}\right)^2 + 1^2\Big)}_{\sum_{\text{gen.}}\frac{3}{5}Y^2} n_g$$

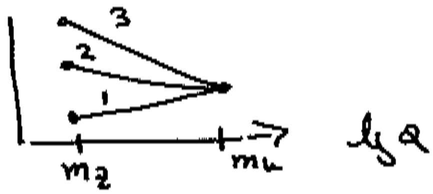
or

$$= \frac{3}{5}\left(\frac{1}{6}+\frac{1}{3}+\frac{4}{3}+\frac{1}{2}+1\right)$$

$$= \frac{3}{5}\,\frac{10}{3} = 2$$

$$b_3 = 11 - \frac{4}{3}n_g$$
$$b_2 = \frac{22}{3} - \frac{4}{3}n_g$$
$$b_1 = -\frac{4}{3}n_g$$

α:



So α_3 gets stronger, α_1 gets weaker for $Q \ll m_U$.

— The Higgs doublet makes a small additional contribution:

b_3: $\quad 0 \qquad 0$

b_2: $\quad -\frac{1}{3}\cdot\frac{1}{2} = -\frac{1}{6}$

b_1: $\quad -\frac{1}{3}\cdot\frac{3}{5}\cdot 2\left(\frac{1}{2}\right)^2 = -\frac{1}{10}$

Can this be made quantitative? The values of α_1 α_2 α_3 are well measured at $Q = m_Z$:

$$\alpha_3(m_Z) \cong \frac{1}{8.3} \qquad \alpha_2 \cong \frac{1}{29.6} \qquad \alpha_1 = \frac{5}{3}\alpha' = \frac{1}{59.1}$$

$$\alpha_3(m_Z) \approx 0.12 \qquad\qquad \left(\alpha' = \frac{1}{98.5}\right)$$

The equations

$$\alpha_i^{-1}(m_Z) = \alpha_i^{-1}(m_U) - \frac{b_i}{2\pi}\log\frac{m_U}{m_Z} \qquad \text{with } \alpha_i^{-1}(m_U) \text{ indep. of } i$$

are 3 equations for 2 unknowns $\alpha(m_U)$, m_U.

So we get 1 prediction, plus the values of $\alpha(m_U)$, m_U.

Take α_1, α_2 as the reference values

$$\log\frac{m_U}{m_Z} = 2\pi\left(\frac{\alpha_1^{-1}-\alpha_2^{-1}}{b_2-b_1}\right) \qquad \alpha_U^{-1} = \frac{b_2\,\alpha_1^{-1}\ \text{[redacted]}\ b_1\alpha_2^{-1}}{b_2-b_1}$$

and

$$\alpha_3^{-1}(m_Z) = \left(\frac{b_2-b_3}{b_2-b_1}\right)\alpha_1^{-1} - \left(\frac{b_1-b_3}{b_2-b_1}\right)\alpha_2^{-1}$$

Note that the formulae for α_3 and m_U depend only on differences of the b_i — $\underline{n_g}$ cancels out. Evaluating for the Standard Model below m_U:

$$\alpha_3^{-1} = 14.2 \qquad \text{or} \qquad \alpha_3(m_Z) \cong 0.07$$

$$\log\frac{m_U}{m_Z} = 25.5 \qquad \text{or} \qquad m_U = 1\times 10^{13}\ \text{GeV}$$

The output value of m_U is very large, but, as we will see, not nearly large enough. The output for α_3 is quite incorrect. So we have the qualitative physics, but not the right quantitative result.

It turns out that we do much better by assuming that the content of the model for $m_Z < Q < m_U$ is the supersymmetric generalization of the Standard Model. Supersymmetry adds

- one left-handed or Majorana fermion in the adjoint rep. for each gauge boson
- one spin-zero complex field for each left-handed fermion
- two Higgs doublets, and associated fermions.

$$b = \frac{11}{3} C(G) - \frac{2}{3} C(G) - \sum_{\text{matter rep.}} \Big(\underbrace{\tfrac{2}{3} C(r)}_{\text{fermion}} + \underbrace{\tfrac{1}{3} C(r)}_{\text{boson}} \Big)$$

$$= 3\, C(G) - \sum_{\text{matter}} C(r)$$

then

$$b_3: \quad 3 \cdot 3 - 2 \cdot \tfrac{1}{2} \cdot 2 \cdot n_g$$

$$b_2: \quad 3 \cdot 2 - \tfrac{1}{2} \cdot (3+1) \cdot n_g - \overbrace{\tfrac{1}{2} \cdot 2}^{\text{Higgs scalars \& fermions.}}$$

$$b_1: \quad - \left(\sum \tfrac{3}{5} Y^2\right) \cdot n_g - \tfrac{3}{5} \cdot 2 \cdot 2 \left(\tfrac{1}{2}\right)^2$$

or

$$b_3 = 9 - 2 n_g$$

$$b_2 = 6 - 2 n_g - 1$$

$$b_1 = - 2 n_g - \tfrac{3}{5}$$

Then $\quad \alpha_3^{-1} = 8.5 \quad$ or $\quad \alpha_3(m_Z) = 0.12$

$\log \frac{m_U}{m_Z} = 33. \quad$ or $\quad m_U = 2\times 10^{16}$ GeV

In general, we get the measured value for $\alpha_3(m_Z)$ if

$$\frac{b_3-b_2}{b_2-b_1} \cong \frac{5}{7}$$

and this is the value in the supersymmetric SM!

Grand unification has interesting implications, of which I will discuss two. Firstly since grand unification puts quarks and leptons into the same multiplets, it predicts new gauge interactions that violate baryon and lepton number. The massive X bosons (p.3) mediate the transitions

$$\left(\begin{array}{c}\bar d\\ \hline L\end{array}\right) \qquad \left(\begin{array}{c|c}\bar u & Q\\ \hline & \bar e\end{array}\right)$$

leading to

$$\Delta\mathcal{L}_{eff} = \frac{g_5^2}{2m_X^2}\,(\bar d)^\dagger\gamma^\mu L\;(\bar u)^\dagger\gamma_\mu Q$$

with all indices explicit, the operator is

$$\underbrace{\epsilon_{ijk}}_{\text{color}}\;\underbrace{\epsilon_{ab}}_{SU(2)}\;(d_R^i)_\alpha\,\varepsilon_{\alpha\beta}\,(u_R^j)_\beta\;Q_L^{ka}\;L^b$$

$$Y = -\tfrac{1}{3} + \tfrac{2}{3} + \tfrac{1}{6} - \tfrac{1}{2} = 0 \;\checkmark$$

this operator mediates processes

$$d\,u\,u \;\to\; e^+ \quad \text{or} \quad u\,d \to \bar{u}\, e^+$$

which leads to proton decay processes such as

$$p \to e^+ \pi^0$$

The rate of this process is

$$\Gamma \sim \alpha_s \frac{1}{M_X{}^4}\, m_p^5 \sim \frac{1}{24}\cdot(1\ \mathrm{GeV})\cdot\left(\frac{m_p}{M_X}\right)^4$$

or

$$\tau \sim 1.6\times 10^{-23}\ \mathrm{sec}\cdot\left(\frac{M_X}{m_p}\right)^4 \sim 5\times10^{-31}\left(\frac{M_X}{m_p}\right)^4\ \mathrm{yr.}$$

Please note that $M_X \sim 10^{13}$ GeV gives $\tau \sim 10^{22}$ yr. That sounds like a long time, but it is not nearly long enough. It is easily possible to do the following experiment: Take a small tank of water, put it in a very well shielded place, watch it very carefully for a year; this would give

$$\tau(p\to e^+\pi^0) > \sim 10^{24}\ \mathrm{yr.}$$

In fact, the current limit is

$$\tau(p \to e^+\pi^0) > 1.6\times 10^{33}\ \mathrm{yr}$$

from the Super-Kamiokande experiment (100 kT of water in an underground cavern). This requires $M_X \gtrsim 10^{16}$ GeV.

In supersymmetric grand unified theories, the most important proton decay modes are actually mediated by heavy particles in the Higgs boson multiplets. The dominant decay is the odd process

$$p \to \nu K^+$$

However, $\tau(p \to \nu K^+) > 6.7 \times 10^{32}$ yr. from super-K.

This number is roughly the same size as the prediction. It is interesting to push the search further.

Grand unified theories can also lead to relations among the Higgs couplings that lead to the quark and lepton masses. In SU(5) grand unification, the Higgs field comes from a 5 of SU(5)

$$\Phi = \begin{pmatrix} \chi \\ \hline \phi \end{pmatrix} \begin{matrix} \} & 3 \text{ of } SU(3) \to \text{must be } \underline{\text{heavy}} \\ \} & \text{Higgs doublet} \end{matrix}$$

the d_R and L come from a $\bar{5}$, and the Q and e_R come from a 10. The SU(5)-invariant coupling

$$\delta\mathcal{L} = -\lambda\, \Psi_{10\,\alpha}^{ab}\, \Psi_{\bar{5}\,a\beta}\, \varepsilon_{\alpha\beta}\, \Phi_b^* + h.c.$$

contains

$$-\lambda\, \bar{d}_R\, \phi^* \cdot Q \;-\; \lambda\, \bar{e}_R\, \phi^* \cdot L$$

which predicts. $\lambda_d = \lambda_\ell$ or $m_d = m_\ell$ in each generation. For τ and b, we have

$$m_b \approx 4.3 \text{ GeV} \qquad m_\tau = 1.77 \text{ GeV}$$

but quark masses receive a QCD correction of about a factor 3 from integrating the effect

from 10^{16} GeV to 1 GeV, so this is not unreasonable. However, all light quark masses receive the same multiplicative correction, so the known ratios:

$$\frac{m_s}{m_d} \sim 15 \qquad \frac{m_\mu}{m_e} = 200$$

are clearly inconsistent with the prediction. It is possible to build models in which small corrections to the Higgs coupling from higher-dimension operators spoil the prediction for s/μ, d/e which keeping the prediction for b/τ.

To go further along either the study of proton decay or that of quark and lepton mass relations, we would have to get much more technical and, in particular, we would have to include the technical properties of supersymmetric GUT's. For a review, see: Stuart Raby, hep-ph/9501349 (1995).

In addition to the ambiguous phenomenological status of grand unification, grand unification has an important purely theoretical difficulty. Let's go back to the Higgs boson GUT multiplet:

$$\Phi = \begin{pmatrix} X \\ \phi \end{pmatrix} \begin{matrix} \} & 3 \text{ of } SU(3) \\ \} & \text{standard Higgs doublet} \end{matrix}$$

X mediates baryon number-violating processes, so it must have a mass $\gtrsim 10^{15}$ GeV. By various stratagems, one can arrange that ϕ has zero mass. However, we do not want $m_\phi^2 = 0$, we want $m_\phi^2 = -\mu^2$ where $\mu \sim 100$ GeV. How could this huge ratio of scales come into the theory? There is no good answer. In addition, the quantum field theory loop corrections to m_ϕ^2 are of order ("the gauge hierarchy problem")

$$\frac{g_5^2}{4\pi} m_U^2 \sim (100 \text{ GeV})^2 \cdot (10^{27}!)$$

So we need to understand why these corrections are absent. There are several answers to this question that have been proposed in the literature; in particular, supersymmetry gives a mechanism. All of these mechanisms require new physics at energies of a few hundred GeV to actually generate $m_\phi^2 = -\mu^2$ at that scale.

Hopefully, the experiments that will be done soon at the CERN LHC will give evidence for this new physics. Those experiments might indicate the presence of supersymmetry or of other new ingredients that have been referred to in this lecture.