Dec. 2

Physics 212 – Statistical Mechanics

The XY Model in 2 Dimensions

There is still a gap in our treatment of magnetic models in 2 dimensions. For $n = 1$ models such as the Ising model, we expect a critical point and a low-temperature ordered phase, similar to the situation in 3 dimensions. For $n \geq 2$, the Mermin-Wagner Theorem forbids a low-temperature phase with nonzero order parameter. We justified this result by studying the XY model in the approximation that Goldstone bosons dominate the fluctuations of the order parameter at low temperature. We showed that, if we make the assumption that the order parameter has an expectation value, this assumption is inconsistent, because the Goldstone boson fluctuations will destroy it.

Let's review this argument. For the XY model, we could represent the spin field as a modulus and a phase

$$\vec{m}(x) = (m(x)\,\cos\theta(x), m(x)\,\sin\theta(x)) \to m(x)\,e^{i\theta(x)} \tag{1}$$

The spin-spin correlation function is given by

$$\langle \vec{m}(x)\cdot\vec{m}(y)\rangle = \mathrm{Re}\,\langle m(x) m^*(y)\rangle \ . \tag{2}$$

Taking the modulus to be fixed by thermodynamics and treating the phase as a Goldstone boson, we can evaluate this expression as

$$\langle \vec{m}(x)\cdot\vec{m}(y)\rangle \approx m_0^2\ \mathrm{Re}\left\langle e^{i\theta(x)}e^{-i\theta(y)}\right\rangle = m_0^2 \exp[G_0(x,y)] \ , \tag{3}$$

where $G_0(x,y)$ is the Green's function with zero mass

$$G_0(x,y) = -\frac{T}{2\pi m_0^2}\log|x-y| \ . \tag{4}$$

Then

$$\langle \vec{m}(x)\cdot\vec{m}(y)\rangle = C\cdot\exp[-\frac{T}{2\pi m_0^2}\log|x-y|] \ , \tag{5}$$

and this implies that

$$\langle \vec{m}(x)\cdot\vec{m}(y)\rangle = C\frac{1}{|x-y|^{T/2\pi m_0^2}} \ . \tag{6}$$

This expectation value tends to 0 as $|x-y| \to \infty$ for any value of the temperature, indicating that there is no magnetic order at any temperature. I than argued that this result provided an upper bound for all spin models with $n \geq 2$, indicating that there is no magnetic order also in the cases of larger n.

However, there is still a problem. The above formula for the spin-spin correlation function corresponds to a power-law decay of correlations and thus an infinite correlation length. However, we know from the convergence of high-temperature expansions that, at very high temperatures, the spin-spin correlation function falls off exponentially

$$\langle \vec{m}(x) \cdot \vec{m}(y) \rangle \sim e^{-|x-y|/\xi(T)} \tag{7}$$

with

$$\xi(T) \to 0 \qquad \text{as} \qquad T \to \infty \ . \tag{8}$$

This seems to differ from the result in (6) and thus to indicate that there must be some sort of phase transition between low and high values of T.

For $n \geq 3$, we saw a way to resolve this difficulty. This comes from the "asymptotic freedom" in the RG flows for these models. However small the original temperature of the model might be, the RG flows carry the models to a high-temperature model at sufficiently large length scales. By solving the RG equation, we found that the effective temperature at the length scale ℓ is given by

$$\frac{T}{1 - \frac{n-2}{2\pi} T \ \log \ell/a} \ , \tag{9}$$

here a is the original lattice spacing and T is the temperature of the original model. If we assume that the high-temperature model has a correlation length of the order of its lattice spacing, $\xi \sim \ell$, then in terms of the original lattice spacing,

$$\xi \sim a \exp\left[\frac{2\pi}{n-2} \frac{1}{T}\right] \ . \tag{10}$$

This is a very large distance as $T \to 0$, but, with this physical picture, the spin-spin correlation function goes to zero exponentially at all nonzero temperatures. Then there is no need for a phase transition.

However, this argument does not apply to the case $n = 2$. For that system, the Goldstone boson is a free field and the behavior (6) is exact at low but finite temperatures. So, in this case, there must be a phase transition at an intermediate temperature. We need to find the physical picture that gives rise to this phase transition.

To study this, let's study the XY model in 2 dimensions in more detail. I consider the XY model on a square lattice. The partition function is

$$Z = \sum_{\vec{S}_i} \ \exp\left[\beta J \ \sum_{i,\nu} \vec{S}_i \cdot \vec{S}_{i+\nu}\right] \ , \tag{11}$$

where $\vec{S}_i$ is a 2-component unit vector. Parametrize $\vec{S}_i$ by

$$\vec{S}_i = (\cos\theta_i, \sin\theta_i) \ . \tag{12}$$

Then

$$Z = \prod_i \int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} \prod_{bonds} \exp\Big[\beta J \cos(\theta_i - \theta_{i+\nu})\Big] \ . \tag{13}$$

If we expand

$$\cos(\theta_i - \theta_{i+\nu}) \to 1 - \frac{1}{2}(\theta_i - \theta_{i+\nu})^2 + \cdots \tag{14}$$

and drop all terms beyond the quadratic order, we seem to prove that the Goldstone boson description is a good approximation. What are we missing?

The answer to this question was given by Michael Kosterlitz and David Thouless in 1973. They claimed that the phase transition in the 2-dimensional XY model results from the thermodynamics of nontrivial topological solutions. This is a fascinating paradigm that extends to a large number of other systems in various dimensions.

Kosterlitz and Thouless introduced their idea more intuitively, but is can be made very explicit by introducing an approximation due to Jacques Villain. Villain proposed the replacement

$$\exp[\beta J \cos(\theta_i - \theta_{i+\nu})] \to \sum_{n_i^\nu=-\infty}^{\infty} \exp[-\frac{1}{2}\beta J \, (\theta_i - \theta_{i+\nu} + 2\pi n_i^\nu)^2] \ . \tag{15}$$

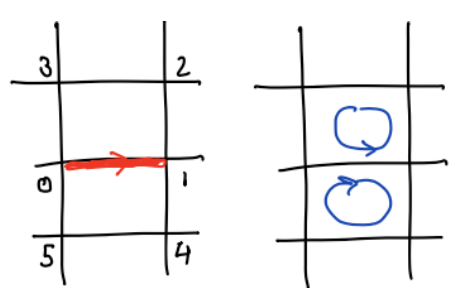
This is really an innocuous change. The two functions have the same behavior near the maximum, and the same periodicity.

To analyze the Villain form of the XY model, let's make a change of variables. On the vertical bonds, we can incorporate all of the integer n_i^y into the angles θ_i

$$\theta_i' = \theta_i + 2\pi n_i^y \tag{16}$$

The new variables θ_i' run from $-\infty$ to ∞. The variables n_i^x remain to be summed over. From here on, I will drop the primes and write the new variables as θ_i.

Let's look more closely at the case in which $n_0^x = 1$ and all other n_i^x are zero. This affects the Boltzmann weight of two elementary squares ("plaquettes") on the lattice



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The weight for these plaquettes is

$$-\frac{1}{2}\beta J\Big\{(\theta_0+2\pi n_0^x-\theta_1)^2+(\theta_1-\theta_2)^2+(\theta_2-\theta_3)^2$$
$$+(\theta_3-\theta_0)^2+(\theta_1-\theta_4)^2+(\theta_4-\theta_5)^2+(\theta_5-\theta_0)^2\Big\} \tag{18}$$

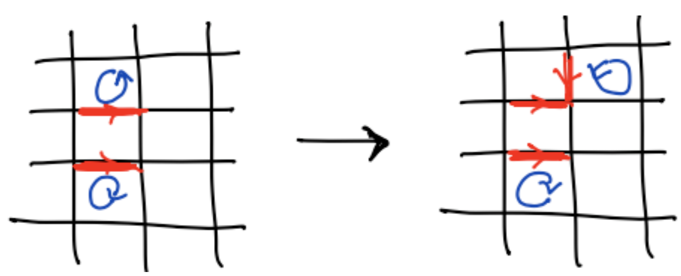
That is, the weight favors a twist of $+2\pi$ in the phases around the upper plaquette and a twist of -2π in the phases around the lower plaquette. The condition that the phase angles twist through $+2\pi$ around an elementaryh plaquette is the boundary condition for a vortex solution in which the phase angles twist through 2π at large distances.

I remind you that the vortex solution has a core whose energy is cut off by the finite lattice spacing and a long-distance part of the energy that is proportional to $\log L$, where L is the size of the system. These features will come into play in just a moment.

The configuration with $n_0^x = 1$ and all other $n_i^x = 0$ actually leads to a dipole, with a positive and negative vortex separated by one lattice spacing. However, it is easy to separate the positive and negative vortices. If we set $n_j^x = 1$ for $j = 0, 1, \ldots, J$, we move the center of the positive vortex to the point $(0, J)$. To move the vortex horizontally, set

$$\theta_{J+\hat{x}} \to \theta_{J+\hat{x}} - 2\pi \ . \tag{19}$$

his effectively restores $n_{J+\hat{x}} = -1$ on a vertical bond and moves the vortex position over by 1 plaquette,



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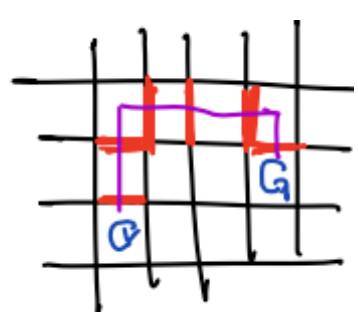
In this way, by rearranging the Villain sum over integers, we can position a the two vortices of a pair at arbitrary positions on the lattice,



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I represent the sequence of nonzero n_j^ν values that give this configuration by a path on the dual lattice that passes through the bonds with $n_j^\nu = \pm 1$. This construction is very similar to the definition of the disorder operator in our discussion of the low-temperature phase of the Ising model. It is also true here that the physics does not depend on the path but only on the locations of the endpoints.

Once we have transformed away the vertical n_j^y, there is one integer degree of freedom for each plaquette. The vorticity in each plaquette is the circulation of n_i^ν about the plaquette. So we can have any number of positive and negative vortices at any positions, as long as the sum of the vorticies is zero.

Let's now work out the interaction of vortices. To do this, we integrate out the θ_i variables, generating an effective interaction of the n_i^ν. We could do this on the lattice, but it will be easier to follow if I go to a continuum description. If a is the lattice spacing, then

$$\sum_i = \int \frac{d^2 z}{a^2} \quad , \qquad (\theta_i - \theta_{i+\nu} + 2\pi n_i^\nu) = -a(\nabla^\nu \theta - \frac{2\pi}{a} n^\nu) \tag{22}$$

Then we need to evaluate the functional integral

$$Z = \int \mathcal{D}\theta \ \exp\Big[-\beta J \int d^2 z \ \Big\{\frac{1}{2}(\vec{\nabla}\theta - \frac{2\pi\vec{n}}{a})^2\Big\}\Big] \tag{23}$$

Taking the measure to be

$$\int \mathcal{D}\theta \ \exp\Big[-\beta J \int d^2 z \ \Big\{\frac{1}{2}(\vec{\nabla}\theta)^2\Big\}\Big] \tag{24}$$

we can write (23) as

$$\exp\Big[-\beta J \int d^2 z \frac{1}{2}(2\pi\vec{n}/a)^2\Big] \cdot \Big\langle \exp\Big[\beta J \int d^2 z \ \vec{\nabla}\theta \cdot 2\pi\vec{n}/a\Big]\Big\rangle \ . \tag{25}$$

The contraction of θ variable under the measure (24) is

$$\langle \theta(z_1)\theta(z_2)\rangle = G(z_1, z_2) = -\frac{T}{2\pi J}\log|z_1 - z_2| \ . \tag{26}$$

Using this contraction, (25) evaluates to

$$\begin{aligned} &\exp\Big[-\beta J \int d^2 z \frac{1}{2}(2\pi\vec{n}/a)^2\Big] \\ &\cdot \exp\Big[(\beta J)^2/2 \int d^2 z_1 d^2 z_2 \ (2\pi\vec{n}_1/a)\cdot\vec{\nabla}_1 \ (2\pi\vec{n}_2/a)\cdot\vec{\nabla}_2 \ G(z_1, z_2)\Big] \end{aligned} \tag{27}$$

We now need to manipulate the expression $\vec{\nabla}_1^a \vec{\nabla}_2^b G(z_1, z_2)$. Since G depends only on $(z_1 - z_2)$, we have

$$\nabla_1^a \nabla_2^b G(z_1 - z_2) = -\nabla_1^a \nabla_1^b G(z_1 - z_2) \ . \tag{28}$$

Now let

$$\widetilde{\nabla}^a = (\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}) \tag{29}$$

and notice that

$$\begin{aligned} -\nabla^a \nabla^b &= -\begin{pmatrix} \partial^2/\partial^2 x & \partial^2/\partial x \partial y \\ \partial^2/\partial x \partial y & \partial^2/\partial^2 y \end{pmatrix} \\ &= -\Big(\delta^{ab}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - \widetilde{\nabla}^a \widetilde{\nabla}^b \\ &= -\delta^{ab} \nabla^2 + \widetilde{\nabla}^a \widetilde{\nabla}^b \end{aligned} \tag{30}$$

The term with ∇_1^2 gives

$$-\nabla_1^2 G(z_1 - z_2) = \frac{1}{\beta J} \delta(z_1 - z_2) \ . \tag{31}$$

Inserting this into the second exponential in (27) gives a term

$$\begin{aligned} &\exp\Big[(\beta J)^2/2 \int d^2 z_1 d^2 z_2 \ (2\pi \vec{n}_1/a) \cdot (2\pi \vec{n}_2/a)(1/\beta J) \ \delta(z_1 - z_2)\Big] \\ &= \exp\Big[(\beta J)/2 \int d^2 z_1 \ (2\pi \vec{n}_1/a) \cdot (2\pi \vec{n}_2/a)\Big] \end{aligned} \tag{32}$$

which cancels leading exponential. Now move one derivative back to z_2 and then integrate by parts under the dz_1 and dz_2 integrals. This gives

$$\exp\Big[-(\beta J)^2/2 \int d^2 z_1 d^2 z_2 \ (2\pi \widetilde{\nabla} \cdot \vec{n}_1/a) \cdot (2\pi \widetilde{\nabla} \cdot \vec{n}_2/a)(1/\beta J)(-\log|z_1 - z_2|\Big] \tag{33}$$

Now,

$$\frac{1}{a} \widetilde{\nabla} \cdot \vec{n}(z) = \frac{1}{a} \vec{\nabla} \times \vec{n}(z) = v(z) \ , \tag{34}$$

where $v(z)$ is the vorticity at the point z. Going back to a discrete representation, we find that the partition function in (23) has been rewritten as

$$Z = \sum_{v_i} e^{+(2\pi \beta J/2) \sum_{i,j} \ v_i v_j \log|z_i - z_j|} \tag{35}$$

where v_i is the (integer) vorticity at the plaquette i. Noting that there are two terms with $v_I v_J$ in the sum, the interaction potential between two vortices is

$$V(z_i - z_j) = -2\pi J \, v_i v_j \ \log|z_i - z_j| \tag{36}$$

The sign of the potential is such that the energy is lower when vortices of opposite sign are close together. Note also that there is a self-energy of a vortex

$$V_{self} = -\frac{2\pi J}{2}\, v_i^2\, \log a \tag{37}$$

where I have used the lattice spacing a to regulate the singularity from the continuum Green's function. This self-energy is positive and thus suppresses the formation of vortices at large β or low T.

Now we come to the beautiful insight of Kosterlitz and Thouless. At very low temperatures, the attraction of positive and negative vortices is very strong, and so it is not possible to have separate positive and negative vortices. Any thermally excited vortices are bound into dipole pairs,

(38)

Let's look inside of one of these pairs as a function of the temperature. Let r be the separation of the positive and negative vortices. The Boltzmann weight of the configuration is

$$\int d^2r\; e^{-\beta(2\pi J \log r)} \tag{39}$$

At low temperature or large β, the exponential falls of rapidly and so the attractive energy dominates and keeps the pair of vortices together. But at higher temperature, or smaller β, the vortices of the pair can separate. In particular, the mean square separation $\langle r^2 \rangle$ involves the integral

$$\left\langle r^2 \right\rangle \sim \int dr r\; r^{-2\pi\beta J}\; r^2 = \int \frac{dr}{r}\; r^{4-2\pi\beta J} \tag{40}$$

and this diverges when

$$4 - 2\pi\beta_c J = 0 \qquad \text{or} \qquad T_c = \frac{\pi}{2} J \tag{41}$$

At this point, the vortices unbind. Above T_c, the system behaves like a vortex plasma.

In 2 dimensions, the Coulomb potential is logarithmic. Then the interaction of vortices is exactly that of charges in 2 dimenions. Thus, the phase transition at T_c is precisely one between an insulating or dielectric phase at low T and a plasma phase at high T. In the low-temperature phase, the vortex pairs are a subdominant effect and we have a power-law falloff of spin correlations due to the Goldstone bosons.

At high T, charges are screened exponentially at the plasma length, and so the spin correlations fall off exponentially, as required.

We can study the transition in more detail by using an RG analysis. Let's put a fixed unit positive vortex at $z = X$ and a fixed negative unit vortex at $z = 0$. At very low temperature, the potential between them will be

$$V(X) = 2\pi J \log(|X|/a) \ . \tag{42}$$

As the temperature increases, vortex pairs will appear, and these dipoles will partially screen the Coulomb potential. I will now use the RG method to account the effect of vortex pairs of all possible sizes, by integrating out one length scale at a time.

The probability of finding a dipole pair with a given position and separation depends on the interaction potential and also on the intrisic Boltzmann weight to add a vortex. This latter parameter is the *fugacity* y; essentially, it is the exponential of the chemical potential for a vortex

$$y = e^{\beta\mu} \tag{43}$$

In the following, I will separate the dependence on the fugacity y from the dependence on the strength of the vortex potential βJ and write a set of coupled RG equations for these parameters.

First, let's work out the RG equation for y. This parameter gets a contribution from the vortex core, with a form proportional to d^2r/a^2 by dimensional analysis, and a contribution from the self-energy computed above due to the Coulomb potential. The full dependence on the lattice spacing is

$$y \sim \frac{1}{a^2}\exp[-\pi\beta J \log a] = \left(\frac{1}{a}\right)^{2-\pi\beta J} \ . \tag{44}$$

If we increase the lattice spacing from a to $a' = \lambda a$, we must incorporate the change in this weight into y. Write (44) as

$$y = y_0\left(\frac{1}{a}\right)^{2-\pi\beta J} = y_0\lambda^{2-\pi\beta J}\left(\frac{1}{a'}\right)^{2-\pi\beta J} = y'\left(\frac{1}{a'}\right)^{2-\pi\beta J} \tag{45}$$

The term in parentheses in the last equality is generated by the new lattice theory, but the statistical weight for a vortex should remain the same. Then we need $y \to y'$, which leads to the RG equation

$$\ell\frac{d}{d\ell}y(\ell) = (2 - \pi\beta J)\ y(\ell) \tag{46}$$

This is a very interesting dependence. It says that y is an irrelevant parameter when $\beta > \beta_c$ or $T < T_c$. In this case, the vortices do not appear in the large-scale effective theory. On the other hand, when $T > T_c$, y is a relevant parameter and the large-scale theory contains a large number of vortices.

Now let's compute the effect of vortex dipoles in screening the Coulomb potential. I will take y to be small and consider the effects of the dipoles one at a time. The expression for the effective potential between two fixed sources, including the effect of one dipole, is

$$e^{-\beta V_{eff}} = e^{-2\pi\beta J \log|X/a|} \left\langle e^{-betaV_{dipole}} \right\rangle \tag{47}$$

where the last factor is a thermal expectation value in the theory of vortex dipoles,

$$\left\langle e^{-\beta V_{dipole}} \right\rangle = \frac{1 + y^2 \int d^2z_+ d^2z_- \; e^{-2\pi\beta J \log|z_+ - z_-|/a + \beta\mathcal{V}} + \cdots}{1 + y^2 \int d^2z_+ d^2z_- \; e^{-2\pi\beta J \log|z_+ - z_-|/a} + \cdots} \; , \tag{48}$$

with $\mathcal{V}$ begin the potential on the fixed vortices at 0 and X due to the dynamical vortex pair,

$$\beta\mathcal{V} = 2\pi\beta J \Big\{ \log\frac{|X - z_+|}{a} - \log\frac{|X - z_-|}{a} - \log\frac{|z_+|}{a} + \log\frac{|z_-|}{a} \Big\} \tag{49}$$

To work out the RG equation, I will focus on vortex dipoles with separation between a and λa, which will be integrated out in one step of the iteration.

To evaluate this expression for the effective potential, change variables to the overall position R and separation r of the dipole,

$$z_+ = R + r/2 \; , \quad z_- = R - r/2 \; , \quad d^2z_+ d^2z_- = d^2R d^2r \; . \tag{50}$$

Expand (48) to the leading nontrivial order in y,

$$\left\langle e^{-\beta V_{dipole}} \right\rangle = 1 + y^2 \int d^2R d^2r \; e^{-2\pi\beta J \log r/a} \Big[\beta\mathcal{V} - 1 \Big] \tag{51}$$

Since the separation r is small, we can expand the expression in brackets in powers of r,

$$\begin{aligned} \Big[\beta\mathcal{V} - 1 \Big] &= \beta\mathcal{V} + \frac{1}{2}(\beta\mathcal{V})^2 + \cdots \\ &= 2\pi\beta J \Big[\log|\frac{X - R - r/2}{X - R + r/2}| - \log|\frac{R - r/2}{R + r/2}| \Big] \\ &\qquad + 2\pi^2(\beta J)^2 \Big[\log|\frac{X - R - r/2}{X - R + r/2}| - \log|\frac{R - r/2}{R + r/2}| \Big]^2 + \cdots . \end{aligned} \tag{52}$$

The first term in the second line is antisymmetric under $\vec{r} \to -\vec{r}$ and so integrates to zero. The second term, for small r, is

$$+2\pi^2(\beta J)^2 \Big[\vec{r} \cdot \vec{\nabla}_R (\log|\frac{X - R}{R}| \Big]^2 \tag{53}$$

Averaging over the direction of $\vec{r}$, this becomes

$$+2\pi^2(\beta J)^2\frac{r^2}{2}\Big[\vec{\nabla}_R(\log|\frac{X-R}{R}|\Big]^2 \tag{54}$$

The expression (54) is integrated over d^2R. This integral can be simplified by integration by parts,

$$\begin{aligned}
&2\pi^2(\beta J)^2\frac{r^2}{2}\int d^2R\Big[\vec{\nabla}_R(\log|\frac{X-R}{R}|\Big]^2\\
&= 2\pi^2(\beta J)^2\frac{r^2}{2}\int d^2R(-\vec{\nabla}_R^2(\log|\frac{X-R}{R}|)\ (\log|X-R|-\log|R|)\\
&= 2\pi^2(\beta J)^2\frac{r^2}{2}\int d^2R(-2\pi\delta(X-R)+2\pi\delta(R))\ (\log|X-R|-\log|R|)\\
&= 2\pi^2(\beta J)^2\frac{r^2}{2}\cdot 2\ (2\pi)\ (\log|X|-\log|0|)
\end{aligned} \tag{55}$$

Finally, replace $\log(0)$ by the regulated expression $\log(a)$. The correction term (48) is then

$$\begin{aligned}
&1+y^2(4\pi^3(\beta J)^2)\int d^2r\ \ r^2e^{-2\pi\beta J\log r/a}\ \ \log|\frac{X}{a}|\\
&1+y^2\ (4\pi^3(\beta J)^2)\ \int_a^{\lambda a} dr\ 2\pi r\ (r/a)^{-2\pi\beta J}\ \log||\frac{X}{a}|\\
&1+y^2\ (8\pi^4(\beta J)^2)\ \int_a^{\lambda a}\frac{dr}{r}\ r^{4-2\pi\beta J}a^{2\pi\beta J}\ \log|\frac{X}{a}|
\end{aligned} \tag{56}$$

This is a correction to the potential between the fixed dipoles given in (42). Then the corrected potential is

$$-\beta V_{eff} = -2\pi(\beta J)'\ \log|X/a|\ , \tag{57}$$

where

$$(\beta J)' = \beta J - 4\pi^3(\beta J)^2y^2\int_a^{\lambda a}\frac{dr}{r}r^{4-2\pi\beta J}a^{2\pi\beta J} \tag{58}$$

We can consider this as an RG evolution equation for (βJ). Differentiating with respect to λ, we find

$$\ell\frac{d}{d\ell}(\beta J) = -4\pi^3(\beta J)^2y^2a^4 \tag{59}$$

We have now found the pair of RG equations for y and (βJ)

$$\begin{aligned}
\ell\frac{d}{d\ell}\ y(\ell)a^2 &= (2-\pi\beta J)\ y(\ell)a^2\\
\ell\frac{d}{d\ell}\ (\beta J) &= -4\pi^3(\beta J)^2\ y^2a^4
\end{aligned} \tag{60}$$

originally derived by Kosterlitz. Note that y has the dimensions of $(\text{length})^{-2}$, so ya^2, using the current value of a, is a dimensionless variable.

In thinking about these equations, it is useful to consider J as fixed and the effective temperature $T(\ell)$ varying with length scale. In this spirit, let's rewrite the (J/T) equation as an equation for $(J/T)^{-1} = T/J$,

$$\begin{aligned} \ell \frac{d}{d\ell}\frac{T}{J} &= -\frac{1}{(\beta J)^2}(-4\pi^3(\beta J)^2 y^2 a^4) \\ &= +4\pi^3 y^2 a^4 \end{aligned} \tag{61}$$

The RG equations (60) have a fixed point at $T/J = \pi/2$, $y = 0$. It is this fixed point that controls the behavior of thermodynamic functions near the Kosterlitz-Thouless phase transition.

To understand the physics in the vicinity of the phase transition, change variables to

$$X = \left(\frac{T}{J} - \frac{\pi}{2}\right) \qquad Y = ya^2 \tag{62}$$

In terms of the new variables, the RG equations are

$$\begin{aligned} \ell \frac{d}{d\ell}\, X &= 4\pi^3\, Y^2 \\ \ell \frac{d}{d\ell}\, Y &= \frac{4}{\pi}\, XY \end{aligned} \tag{63}$$

Our approximations to derive these questions are valid when Y is small and $X = T/J$ is close to $\pi/2$.

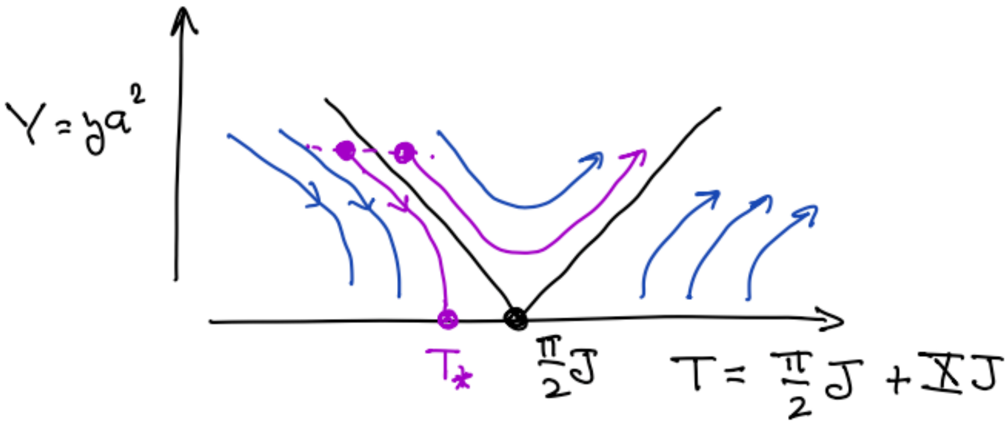
Notice that

$$\ell \frac{d}{d\ell}(X^2 - \pi^4 Y^2) = 2X \cdot 4\pi^3\, Y^2 - 2Y \cdot \pi^4 \cdot \frac{4}{\pi} XY = 0\ . \tag{64}$$

Then the RG flows are along hyperboloids,

$$X^2 - \pi^4 Y^2 = \text{const.} \tag{65}$$

Here is the flow diagram:



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At low T, the flows go into the axis $Y = 0$ Let T_* be the temperature corresponding to the intersection of the hyperboloid with the $Y = 0$ axis. The large-scale theory is then a theory with no vortices and $T = T_*$. The spin-spin correlation function behaves as

$$\left\langle \vec{S}(x) \cdot \vec{S}(y) \right\rangle \sim \frac{1}{|x-y|^{T_*/2\pi J}} \tag{67}$$

We refer to this as a *line of fixed points*, since the exponent in this power law varies continuously as a function of the underlying parameters.

If we start with finite y, the critical temperature of the model is given by the condition

$$X = -\pi^2 Y \tag{68}$$

or

$$T_c(y) = J\Big[\frac{\pi}{2} - \pi^2 y a^2\Big] \tag{69}$$

Slightly below this temperature, at a temperature

$$T = T_c - \Delta = J\Big[\frac{\pi}{2} - \pi^2 y a^2 - \frac{\Delta}{J}\Big] \ , \tag{70}$$

the asymptotic power law is given by

$$\begin{aligned} T_*(y) &= \frac{\pi}{2}J - J\Big[(\pi^2 y a^2 + \frac{\Delta}{J})^2 - \pi^4 y^2 a^4\Big]^{1/2} \\ &= \frac{\pi}{2}J - \Big[2\pi^2 y a^2 J(T_c(y) - T)\Big]^{1/2} \ . \end{aligned} \tag{71}$$

What if we start just slightly on the high-temperature side of this line? Let

$$T = T_c + \Delta = J\Big[\frac{\pi}{2} - \pi^2 y a^2 + \frac{\Delta}{J}\Big] \ , \tag{72}$$

The RG flow takes us to the vicinity of the fixed point and then out the other side to a plasma phase where $y(\ell)$ is large. We can estimate the correlation length as the value of ℓ at which X and Y become of order 1. To find this, first note that the initial values of X and Y are

$$X_0 = (-\pi^2 y a^2 + \frac{\Delta}{J}) \qquad Y_0 = y^2 a^4 \tag{73}$$

Then the hyperboloid is

$$(X^2 - \pi^4 Y^2) = -2\pi^2 y a^2 \frac{\Delta}{J} \tag{74}$$

With this formula, we vcan solve for the later values of Y from the formula for the hyperboloid

$$Y^2 = \frac{1}{\pi^4}(X^2 + 2\pi^2 y a^2 \frac{\Delta}{J}) \ . \tag{75}$$

Plugging this into the RG equation for X equation gives

$$\ell \frac{d}{d\ell} X = \frac{4}{\pi}(X^2 + +2\pi^2 y a^2 \frac{\Delta}{J}) \tag{76}$$

Now we can integrate this equation,

$$\int \frac{dX}{X^2 + 2\pi^2 y a^2 \Delta/J} = \frac{4}{\pi^2} \int d\log \ell \ . \tag{77}$$

The result is

$$\frac{1}{\sqrt{2\pi y a^2 \Delta/J}} \tan^{-1} \frac{X}{\sqrt{2\pi y a^2 \Delta/J}} = \frac{4}{\pi} \log \ell/a \tag{78}$$

or

$$\ell = a \exp\Big[\frac{\pi/4}{\sqrt{2\pi y a^2 \Delta/J}} \tan^{-1} \frac{X}{\sqrt{2\pi y a^2 \Delta/J}}\Big] \tag{79}$$

Now set $\xi = \ell$ for $X \gg (T - T_c(y))$. The $\tan^{-1}$ will be close to 1. This gives

$$\xi(y, \Delta) = C \cdot \exp\Big[\Big(\frac{\pi/8}{y a^2/J}\Big)^{1/2} \frac{1}{\sqrt{T - T_c}}\Big] \tag{80}$$

This is not a power law, and, in fact, the expression grows faster than any power law as $T \to T_c$. This adds to our catalog of strange behaviors of the thermodynamic functions in the vicinity of a critical point.