Nov. 2

Physics 212 – Statistical Mechanics

Electromagnetism and Superconductivity

In the previous lecture, I described the microscopic basis of superconductivity through fermion pair condensation. This allowed us to write the Landau free energy describing the phase transition to the superconducting state. However, there is another ingredient that we need to add to the story. The fermion pair condensate has electric charge and couples to the electromagnetic field. This adds new elements that form an essential part of the behavior of superconductors. I will describe some of these effects in this lecture.

The Cooper pair quantum field $\Phi(x)$ defined in the previous lecture destroys electric charge $Q = -2$. In external fields, the Cooper pairs might have nontrivial wavefunctions $\psi(x)$. Using a basis for these wavefunctions, the quantum field $\Phi(x)$ would take the form

$$\Phi(x) = \sum_n \psi_n(x)\; b_n \tag{1}$$

where b_n is an annihilation operator that annihilates a Cooper pair in the wavefunction $\psi_n(x)$. These wavefunctions satisfy a Schrödinger equation with electric change Q, and this property carries over to the quantum field. Thus, in accord with the usual prescription from quantum mechanics, we must modify the derivatives in the Landau free energy to covariant derivatives

$$\vec{\nabla} \to \vec{D} = \vec{\nabla} - i\frac{e}{c}Q\vec{A}(x)\ . \tag{2}$$

This gives the Landau-Ginzburg free energy for a superconductor

$$G[\Phi] = \int d^3x \Big\{ \frac{1}{2m}|(\vec{\nabla} - i\frac{e}{c}Q\vec{A})\Phi|^2 + \frac{1}{2}a(T-T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \Big\}\ . \tag{3}$$

This change has an important consequence. If $\Phi(x)$ acquires a constant thermodynamic expectation value Φ_0, then the Landau-Ginzburg free energy has a new term that depends on the electromagnetic vector potential

$$\int d^3x\, \left|\frac{Qe}{c}\Phi_0\right|^2 |\vec{A}|^2 \tag{4}$$

This term gives a new contribution to Maxwell's equations in the presence of a superconductor.

For a field with charge $Q = -2e$, the electric current is

$$\vec{J} = \frac{-i}{2m}\frac{Qe}{c}(\Phi^* \vec{D}\Phi - (\vec{D}\Phi)^*\Phi) \tag{5}$$

Notice that the ordinary derivatives in the current operator are replaced by covariant derivatives. For a constant $\Phi(x) = \Phi_0$, this formula reads

$$\vec{J} = \frac{Q^2 e^2}{c^2}\Phi_0^2 |\vec{A}|^2 \tag{6}$$

This equation is called the *London equation*, after Fritz London, who explored the theory of quantum fluids in the 1930's. Let's plus this expression into Maxwell's equations

$$\begin{aligned} -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} &= \frac{4\pi}{c}\vec{J} \\ \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} &= 0 \end{aligned} \tag{7}$$

Then

$$\begin{aligned} 0 &= \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \vec{\nabla} \times \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \\ &= \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \vec{\nabla} \times \vec{\nabla} \times \vec{B} + \frac{4\pi}{c}\left(\frac{Qe}{c}\right)|\Phi_0|^2 \vec{\nabla} \times \vec{A} \end{aligned} \tag{8}$$

Since

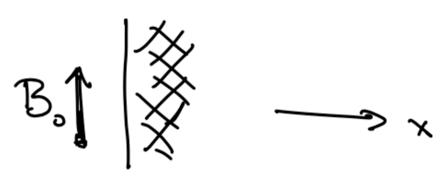
$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} \tag{9}$$

and $\vec{\nabla} \cdot B = 0$, the equation becomes

$$0 = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{4\pi}{mc}\left(\frac{Qe}{c}\right)^2 |\Phi_0|^2\right] \vec{B}\ . \tag{10}$$

This is no longer a massless wave equation for electromagnetic waves and photons. Instead, it is a *massive* Klein-Gordon equation.

An implication of this equation is that if $\vec{B}$ takes a fixed static value B_0 at the boundary of a superconductor



(11)

then $\vec{B}$ falls off in the interior as

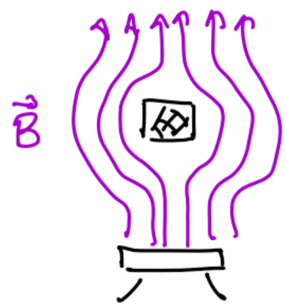
$$B(x) = B_0 e^{-x/\lambda} \tag{12}$$

where

$$\lambda = \left[\frac{4\pi Q^2 e^2}{mc^3} \Phi_0^2 \right]^{-1/2} \tag{13}$$

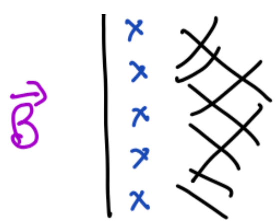
The length λ is called the *penetration depth*. Notice that we now have two basic lengths connected to superconductivity, the correlation length of the symmetry-breaking condensate ξ and the penetration depth λ. I will discuss the interplay of these two lengths in a moment.

The result here that a superconductor expels magnetic flux is called the *Meissner effect*. This is the basis for magnetic levitation of a superconducting block, which you may have seen in the laboratory.



(14)

Physically, the magnetic field induces a supercurrent in the skin of the superconductor, and this shields the interior



(15)

The converse of the statement that superconductors exclude magnetic flux is that magnetic fields destabilize superconductivity. There is a *critical field* above which a metal cannot remain superconducting. I will now compute this field using the Landau-Ginzburg theory. In this derivation, it is useful to use $\vec{H}$ for the externally applied field. If $\vec{M}$ is the internal magnetization,

$$\vec{B} = \vec{H} + 4\pi \vec{M} \tag{16}$$

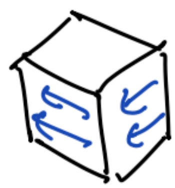
If the metal is in its normal state, the Gibbs free energy is the magnetic field energy

$$\int d^3x \, \frac{c}{8\pi} H^2 \ . \tag{17}$$

If the metal is in its superconducting state, there is the usual contribution to the free energy from spontaneous symmetry breaking

$$\int d^3x \Big\{ -\frac{1}{4} \frac{a^2(Tc - T)^2}{b} \Big\} \tag{18}$$

But also, the supercurrents in the walls



(19)

lead to a magnetic moment that cancels the applied field

$$\vec{M} = -\frac{c\vec{H}}{4\pi} \tag{20}$$

and this contributes to the free energy

$$G_M = \int d^3x \; (-\vec{H} \cdot \vec{M}) = \int d^3x \; (\frac{+cH^2}{4\pi}) \tag{21}$$

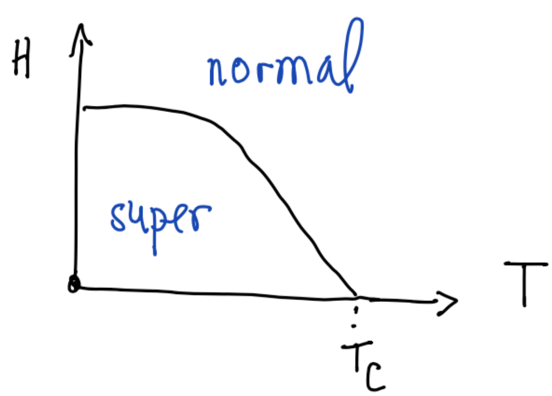
Balancing these effects, the free energy difference per unit volume of the superconductor is

$$\Delta G = \int d^3x \Big\{ -\frac{1}{4} \frac{a^2(T_c - T)^2}{b} + \frac{cH^2}{8\pi} \Big\} \tag{22}$$

Thus, the normal metal is favored when H is sufficiently large. The *critical field* is

$$H_c = \left[\frac{2\pi a^2}{bc} \right]^{1/2} (T_c - T) \; . \tag{23}$$

Note that, near the critical point, there is a linear relation between H_c and $(T_c - T)$. Then the phase diagram of a superconductor will have the form



(24)

When one crosses the phase boundary, there is a discontinuous transition from the superconducting state to the normal metal.

I would now like to go back and discuss the theory of the generation of the B field or photon mass a little further. I have emphasized already that quantum mechanics has a symmetry under phase rotation of the wavefunction

$$\Phi(x) \to e^{i\alpha}\Phi(x) \tag{25}$$

However, when the Schrödinger wavefunction is coupled to an electromagnetic field, this symmetry is extended to a symmetry that depends on a function $\alpha(x)$. Explicitly,

$$\Phi(x) \to e^{iQ\alpha(x)}\Phi(x) \qquad \vec{A} \to \vec{A}(x) + \frac{c}{e}\vec{\nabla}\alpha(x) \tag{26}$$

where Q is the electric charge of the quanta, is now a good symmetry of the Schrödinger equation for any function $\alpha(x)$. Under this transformation, the covariant derivative transforms as

$$\begin{aligned} \vec{D}\Phi &= (\vec{\nabla} - i\frac{e}{c}Q\vec{A})\Phi \\ &\to (\vec{\nabla} - i\frac{e}{c}Q\vec{A} - iQ\vec{\nabla}\alpha)e^{iQ\alpha}\Phi \\ &= e^{iQ\alpha}((\vec{\nabla} + iQ\vec{\nabla}\alpha - i\frac{e}{c}Q\vec{A} - iQ\vec{\nabla}\alpha)\Phi \end{aligned} \tag{27}$$

or, finally,

$$\vec{D}\Phi(x) \to e^{iQ\alpha(x)}\vec{D}\Phi(x) \ . \tag{28}$$

Then the expression

$$\int d^3x \frac{1}{2m}|\vec{D}\Phi|^2 \tag{29}$$

is invariant, and, indeed, the entire Landau-Ginzburg free energy is invariant under this symmetry. As a consequence, the electric current (5) is also invariant. Notice that the invariance of $\vec{J}$ requires the presence of a term in the current proportional to $\vec{A}$, as seen above.

The symmetry (26) is called *local gauge invariance* based on the $U(1)$ symmetry. You have seen it before in your undergraduate electrodynamics course as the transformation

$$\vec{A} \to \vec{A} + \vec{\nabla}\alpha \qquad A^0 \to A^0 - \frac{1}{c}\frac{\partial}{\partial t}\alpha \ , \tag{30}$$

which is a symmetry of Maxwell's equations. We can turn this around and state that Maxwell's equations are the unique linear equations for the vector potential $A^\mu(x)$ that respect local gauge invariance. That is, it is possible to take the local gauge invariance as the axiom and use it to derive the field equation for A^μ. This logic turns out to generalize and actually it gives us the theories of the other known fundamental interactions– the strong and weak interactions, and gravity.

The transformation

$$\vec{A} \to \vec{A} + \frac{c}{e}\vec{\nabla}\alpha \tag{31}$$

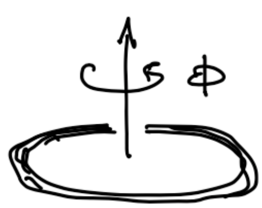
would seem to prohibit a mass term for the photon

$$\int d^3x \; \frac{1}{2} m_A^2 |\vec{A}|^2 \; . \tag{32}$$

However, we have just seen that, in a system in which the $U(1)$ symmetry is spontaneously broken, the photon does acquire a mass. It turns out to be true more generally, in relativistic quantum field theory, that we can only give a mass to a vector field in a theory with *local gauge invariance* in which the local gauge symmetry is *spontaneously broken*.

A way to understand the importance of local gauge symmetry in this phenomenon is the following: The photons described by Maxwell's equations have only 2 degrees of freedom, the two orthogonal states of linear polarization. If we add a complex scalar field, we now have 4 degrees of freedom. Now let the $U(1)$ symmetry be spontaneously broken. In that case, we can use the local gauge freedom to remove the phase of $\Phi(x)$ and make it a real number. Then the Goldstone boson degree of freedom in $\Phi(x)$ disappears from the theory. In return, we find an additional degree of freedom in electromagnetism. Though the massless photon has only 2 polarization states, a massive spin-1 particle must have has 3 possible polarization states, corresponding to the 3 directions of vectors in its rest frame. This transmutation of degrees of freedom is necessary to give a spin-1 particle a mass, and spontaneous symmetry breaking accomplishes it. This phenomenon is called the *Higgs mechanism* (an abbreviation for credit to Peter Higgs, Francois Englert, Robert Brout, Gerald Guralnik, Carl Hagen, and Thomas Kibble). The working of this mechanism in superconductivity was understood shortly after the BCS paper by Yoichiro Nambu and Philip Anderson.

Note that, although, we can use gauge invariance to remove the phase of $\Phi(x)$, we can also choose the gauge parameter to make this phase have any variation convenient to our purposes. (The Goldstone fluctuation in the phase is still gone.) For example, if we have a superconduting wire, we can have the phase vary around the wire



(33)

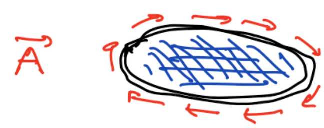
as long as it comes back periodically to the original value $(+2\pi n)$ when we traverse the loop. This corresponds to a supercurrent flowing around the loop of wire. Again note that a situation with a nonzero twist is topologically stable and cannot be smoothly deformed away. This supercurrent will circulate forever (rather, until there is a very large, very rare, thermal fluctuation) without resistance. We can also transform away the phase, but this gives a nonzero value of $\vec{A}$ around the wire, satisfying

$$\vec{A} = \frac{-\hat{\phi}}{r} \frac{n}{Qe/c} \tag{34}$$

Because of the $\vec{A}$ term in the current, there is still a current flowing in the wire in this gauge. Using

$$\int d^2\vec{s} \cdot \vec{B} \oint d\vec{x} \cdot \vec{A} = -\frac{2\pi n\hbar c}{Qe} \tag{35}$$

we find that the wire encloses a quantized magnetic flux,



(36)

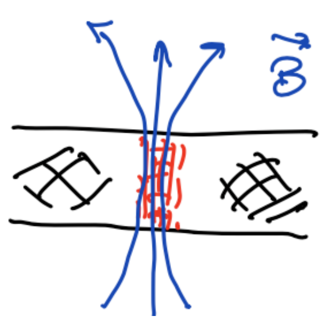
$$\int d^2\vec{s} \cdot \vec{B} = \frac{2\pi\hbar c}{Qe} \cdot n \tag{37}$$

The flux quantum reflects the charge of the order parameter. Flux quantization in a superconducting loop was first observed here at Stanford in 1961 by William Fairbank and Bascom Deaver. They showed that the observed quantization corresponds to the Cooper pair charge of $|Q| = 2e$.

Let me now come back to the relation of the two characteristic lengths in a superconductor, ξ and λ. There are two different possible relations,

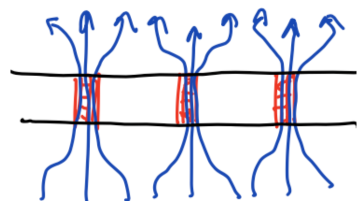
$$\begin{aligned} \text{Type I} &: \quad \xi > \lambda \\ \text{Type II} &: \quad \xi < \lambda \end{aligned} \tag{38}$$

In both cases, in principle, we can try to thread a quantized magnetic flux through the superconductor by driving a small tube of material normal,



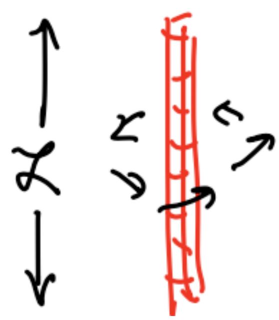
(39)

In a Type I superconductor, such small flux tubes are not sustainable. The radius of the normal region will be of order ξ, a large distance, so the free energy cost of making the tube will be large. The natural size of the region filled with flux will be much smaller. If we have two such flux tubes, they will attract and merge. For a macroscopic total flux, a large region of the superconductor will go normal, and we will have the phase diagram shown in (24). However, in a Type II superconductor, another outcome is possible. The superconductor can break down in small tubes, of width ξ, with the magnetic field extending far outside. These tubes will repel one another, leading to an array of tubes, each carrying one flux quantum,



(40)

An individual magnetic tube in a Type II superconductor has an interesting structure, discovered by Alexei Abrikosov and called the *Abrikosov flux tube.* We can solve for this structure in the limit $\xi \ll \lambda$. Then the superconducting order parameter can be taken to have the form of a vortex solution



(41)

That is,

$$\Phi = f(r)\Phi_0 e^{i\phi} \tag{42}$$

with the function $f(r)$ going exponentially rapidly to 1 for $r > \xi$. I will write $\mathcal{L}$ for the length of the vortex, as shown in (41). Let's examine the situation first at large values of r. Here

$$\vec{\nabla}\Phi = \frac{\hat{\phi}}{r}\frac{\partial}{\partial\phi}\Phi = i\frac{\hat{\phi}}{r}\Phi_0 \tag{43}$$

This by itself gives

$$G[\Phi] = \int d^3x \, \frac{1}{2m}|\vec{\nabla}\Phi|^2 + \cdots \tag{44}$$

By itself, this would be a problem, since then the kinetic term contribution to the Gibbs free energy is logarithmically divergent at large r,

$$G[\Phi] = \mathcal{L} \cdot \frac{\pi}{m}\int dr \, r \, \frac{\Phi_0^2}{r^2} + \cdots \tag{45}$$

However, we can lower the energy of this solution by turning on an $\vec{A}$ field with the large-distance behavior

$$\vec{A} \to +\frac{\hat{\phi}}{Qe/c}\,\frac{1}{r} \tag{46}$$

Then, for large r,

$$\vec{D}\Phi = \vec{\nabla}\Phi - i\frac{Qe}{c}\vec{A}\Phi = \hat{\phi}\Phi_0 \cdot [\frac{i}{r} - \frac{i}{r}] \tag{47}$$

and the dangerous term for large r cancels. Note also that $\vec{A}$ is "pure gauge", that is, it is a gradient that can be removed by a gauge transformation. For such an azimuthal field, the curl is

$$\vec{\nabla} \times \vec{V} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}(rv_\phi) \ . \tag{48}$$

o, for $A_\phi \sim 1/r$, $\vec{\nabla} \times \vec{A} = 0$ and there is no magnetic field at large r. As we will see, the magnetic field and the density of Gibbs free energy falls exponentially in the distance of the penetration depth. The integral of $\vec{A}$ around a large circle is

$$\oint d\vec{x} \cdot \vec{A} = \frac{2\pi\hbar c}{Qe} \tag{49}$$

corresponding to one flux quantum.

To find the explicit form of the magnetic field, we can work from the variational equation for the Gibbs free energy coupled to electromagnetism. Actually, though, it is easier to use the London equation and the associated version of Maxwell's equation that I wrote earlier. In this static situation,

$$\Big[-\nabla^2 + \frac{1}{\lambda^2}\Big]\vec{B} = 0 \tag{50}$$

Let's look for a solution that is a rotationally symmetric magnetic field in the $\hat{z}$ direction

$$\vec{B} = B_z(r)\,\hat{z} \tag{51}$$

This obeys the equation

$$\Big[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{\lambda^2}\Big]B_z(r) = 0 \tag{52}$$

The solution is the modified Bessel function

$$B_z(r) = C \cdot K_0(r/\lambda) \ . \tag{53}$$

Using the Bessel function relations

$$\frac{1}{z}\frac{d}{dz}(zK_1(z)) = -K_0(z) \tag{54}$$

we can see that the solution (53) is the curl of an $\vec{A}$ field of the form

$$\vec{A} \propto \hat{\phi}\Big(-\frac{1}{\lambda}K_1(r/\lambda)\Big) \tag{55}$$

plus any other $\vec{A}$ field that is curl free or pure gauge. As $r \to 0$,

$$K_1(r/\lambda) = \frac{\lambda}{r} + \cdots \ , \tag{56}$$

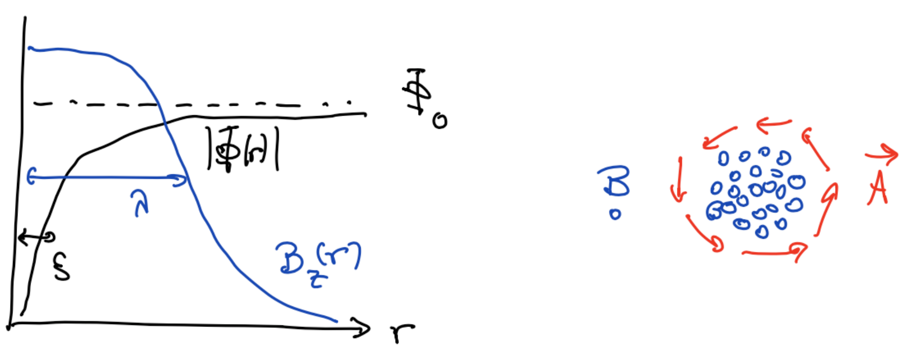
and $K_1(r/\lambda)$ goes to zero exponentially as $r \to \infty$. Then the combination

$$\vec{A} = \frac{c}{Qe}\hat{\phi}\left[\frac{1}{r} - \frac{1}{\lambda}K_1(r/\lambda)\right] \tag{57}$$

satisfies all of our requirements: The $\vec{A}$ field is nonsingular as $r \to 0$ and the form (46) as $r \to \infty$, and its curl satisfies (50). The corresponding magnetic field is

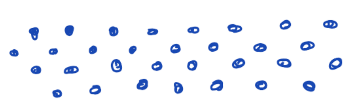
$$\vec{B} = \hat{z}\Big(\frac{\hbar c}{Qe}\frac{1}{\lambda^2}K_0(r/\lambda)\Big) \ . \tag{58}$$

$K_0(r/\lambda)$ still has a logarithmic singularity, but this is cut in the core of the flux tube where $r < \xi$ and the approximation $|\Phi(r)| = \Phi_0$ no longer holds. The solution (58) obeys the flux quantization. Here is a sketch of the full solution



(59)

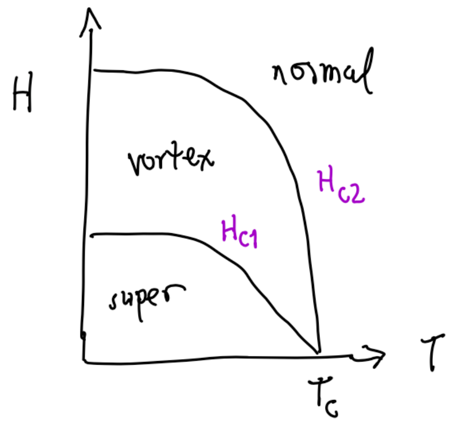
Abrikosov realized that these vortices produce a new thermodynamic phase in the phase diagram of H versus T. As a certain magnetic field H_{c1}, magnetic field can penetrate a Type II superconductor without breaking down the overall superconductivity. The field threads through in an array of vortices, as shown in (40). This is called the *Abrikosov vortex state*. In this phase, magnetism and superconductivity coexist. The most stable configuration of vortices is a hexagonal lattice



(60)

Superconducting magnets, which can sustain very high fields in special materials such as niobium, operate in the vortex phase.

At a higher field H_{c2}, the vortices become too dense and the superconductivity is destroyed. The full phase diagram of a Type II superconductor is then



(61)

You can find much more about the physics of these vortices in the book by Michael Tinkham, *Introduction to Superconductivity.*