Oct. 26

Physics 212 – Statistical Mechanics

Superfluidity

Some well-known examples of phase transitions in condensed matter physics are the transitions to superfluidity and superconductivity. In this lecture and the next two, I will explain how to describe these phenomena and how to fit them into Landau theory.

Both superfluidity and superconductivity are essentially quantum mechanical phenomena. They occur at very low temperatures, typically within a few degrees of absolute zero. Both were first created by Heike Kamerlingh Onnes, who was the first person to create temperatures low enough to liquify helium in 1908. Kamerlingh Onnes actually observed superconductivity, the flow of electricity without resistance, in 1911. He also observed the λ transition in helium, a sharp peak in the specific heat at 2.2°K associated with the superfluid transition, but superflow, fluid flow with zero viscosity, was not actually observed until experiments by Piotr Kapitsa in 1937. (Kamerlingh Onnes may have observed "superleaks".) In the 1940's, superfluidity was explored in detail through theoretical work by Lev Landau in close interaction with Kapitsa and the members of his laboratory.

If you have studied the free boson gas, you know about the phenomenon of Bose-Einstein condensation. At temperator T and chemical potential μ, the occupation number of energy levels of a boson gas is

$$\frac{1}{e^{\beta(E_k-\mu)}-1} \ . \tag{1}$$

For a gas of nonrelativistic particles with energy $E_k = k^2/2m$, the total number density is

$$\rho = \frac{N}{V} = \int \frac{d^3k}{(2\pi)^3} \, \frac{1}{e^{\beta(E_k-\mu)}-1} \ . \tag{2}$$

In practice, we fix μ to give the correct particle density for the system we wish to describe. However, we must have $\mu \leq 0$ to avoid a non-integrable singularity in the integral. The maximum particle number described by this formula is then the value at $\mu = 0$,

$$\rho = \frac{N}{V} = \int \frac{d^3k}{(2\pi)^3} \, \frac{1}{e^{\beta E_k}-1} \ . \tag{3}$$

or

$$\rho_*(T) = \left(\frac{mT}{2\pi}\right)^{3/2} \zeta(\frac{3}{2}) \ , \tag{4}$$

with $\zeta(z)$ the Riemann zeta function. As $T \to 0$, we always reach a temperature T_c such that $\rho_* = \rho_*(T_c)$. Below this temperature, the extra atoms, whose number is

$$N_c = V \cdot (\rho - \rho_*(T)) \tag{5}$$

all sit in the $\vec{k} = 0$ energy eigenstate. Then, this single quantum state becomes macroscopically occupied. It can be shown that the specific heat

$$C = \frac{\partial E}{\partial T}\bigg|_{V,N} \tag{6}$$

has a discontinuous slope at this critical temperature.

The physics of a gas of strongly interacting bosons such as He^4 atoms is more subtle. Helium atoms are not free particles; they have strong hard-core repulsive interactions. So the zero-temperature state of liquid He^4 is not a simple Bose condensate. Instead, it is a highly correlated wavefunction. To describe the properties of this state, we need a different set of principles. In this lecture, I will build these up using the idea of a *quantum field*.

Look back at the first lecture for the definition of a quantum field. There I introduced the creation and annihilation operators that have the commutation relations.

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{q}) \tag{7}$$

These describe free particles, but we can also use them to build up the quantum Hamiltonian for a system of interacting particles. By Fourier transforming, we convert states in momentum space to states in position space. Hence, the quantum fields

$$\Phi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} a_p^\dagger \qquad \Phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} a_p \tag{8}$$

are operators on Fock space that create and destroy particles at the position $\vec{x}$.

Notice that Φ and $\Phi^\dagger$ change the number of particles in the system. Thus, we cannot simultaneously diagonalize the operator for the total number of particles

$$N = \int \frac{d^3p}{(2\pi)^3}\, a_p^\dagger a_p \tag{9}$$

and the quantum field operators. Typically, we consider systems with a fixed number of particles. However, the grand canonical ensemble blurs this a little, since we consider states with different numbers of particles in the same ensemble, controlled by the chemical potential μ. In particular, the Bose condensate is described as a state with an indefinite number of particles in the $\vec{p} = 0$ wavefunction. However, this is a classical averaging, and we could just as well talk about a state with a fixed but macroscopic number of particles in the $\vec{p} = 0$ state. The difference between these states is small according to $1/\sqrt{N}$.

However, when we discuss interacting particles, it is advantageous to introduce coherence among particles at different spatial positions. One of the terms in the Hamiltonian is the interatomic potential

$$\Delta\mathcal{H} = \int d^3x d^3y \; V(x-y) \; \Phi^\dagger(x)\Phi(x)\,\Phi^\dagger(y)\Phi(y) \tag{10}$$

To diagonalize a Hamiltonian with such an interaction it is useful to make the various atoms that may exist at $\vec{x} = 0$ coherent with one another. This produces a new type of quantum state.

I would now like to make the following proposal: A *superfluid* is a system in which the quantum field $\Phi(\vec{x})$ has a thermodynamic expectation value. Thus, we write in the ground state

$$\langle \Phi(\vec{x}) \rangle = \sqrt{\rho_s} \tag{11}$$

where ρ_s is the *superfluid density* in atoms/cm^3. In general, ρ is a function of temperature. In the rest of this lecture, I will show that this assumption, combined with our Landau theory description of states of spontaneously broken symmetry, will lead to the properties of a superfluid of having fluid flow with zero viscosity.

Let's recognize that the state (11) is a state of spontaneously broken symmetry. First of all, since Φ changes the number of particles, particle number is no longer a good quantum number in this state. More generally, in quantum mechanics, a single-particle wavefunction give unchanged physics when its overall phase is rotated,

$$\psi(x) \to e^{i\alpha}\psi(x) \tag{12}$$

The phase rotation symmetry is a fundamental part of quantum mechanics, but it also is the symmetry that, for quantum fields, gives rise to particle number conservation via Noether's Theorem. The expectation value (11) breaks that symmetry.

Notice that (12) is a $U(1)$ symmetry, so this symmetry-breaking is the same one that we saw in the XY ferromagnet, and so the two systems share the same Landau theory description. Then we can write an approximate Gibbs free energy for superfluidity,

$$G[\Phi] = \int d^3x \left\{ \frac{1}{2m}|\vec{\nabla}\Phi|^2 + \frac{1}{2}a(T-T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right\} \tag{13}$$

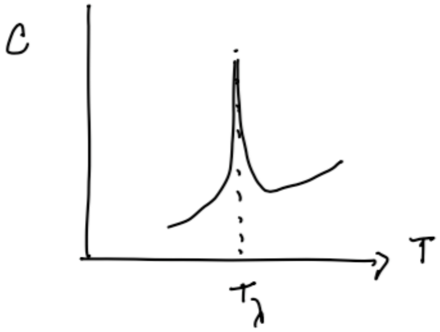
This expression will predict that the expectation value of Φ turns on at $T = T_c$ according to

$$\langle \Phi \rangle \sim [T_c - T]^{1/2} \tag{14}$$

giving for the superfluid density

$$\rho_s(T) \sim (T_c - T) \tag{15}$$

which is the same law followed by the occupation number of the $\vec{p} = 0$ state in Bose condensation. I note, though, that in a real superfluid such as He^4 the power law is slightly different. Also the specific heat anomaly at T_c is not a discontinuity but rather a singularity that is close to logarithmic,



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This is why the superfluid transition is called the *lambda transition.*

In an XY magnet, the only low-lying excitations are the Goldstone bosons. The same thing will be true for a superfluid. Let's now study the consequences of this. First, we must introduce the supercurrent operator. In quantum mechanics, a single-particle wavefunction has an associated current operator,

$$\vec{j} = \frac{-i}{2m}(\psi^\dagger\, \vec{\nabla}\psi - \vec{\nabla}\psi^\dagger\, \psi) \tag{17}$$

For a wavefunction

$$\psi \sim e^{i\vec{k}\cdot\vec{x}} \tag{18}$$

this operator evaluates to

$$\vec{j} = \frac{\vec{k}}{m}\ , \tag{19}$$

the particle velocity. Similarly, if we write

$$\vec{j} = \frac{-i}{2m}(\Phi^\dagger\, \vec{\nabla}\Phi - \vec{\nabla}\Phi^\dagger\, \Phi) \tag{20}$$

this operator will represent the superflow current. The current is zero in the ground state (11), but if we add a slow twist along the Goldstone boson direction,

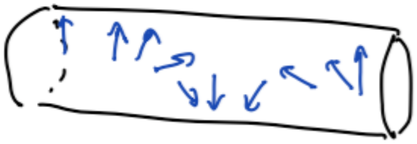
$$\langle \Phi(\vec{x}) \rangle = \sqrt{\rho_s}\ e^{i\vec{k}\cdot\vec{x}} \tag{21}$$

then in this state

$$\vec{J} = \rho_s \frac{\vec{k}}{m}\ , \tag{22}$$

the particle flow in atoms/cm^2-sec.

The phase of Φ is a macroscopic quantity. It cannot relax to zero on its own, because there are no other low-lying states to which it can give up its energy. As an illustration of this, imagine that the phase of Φ twists down a pipe,



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If we connect the ends of the pipe to form a torus, the phase advance through the pipe is a topological invariant and cannot be changed by continuous changes in the local thermodynamic state. Then the phase advance cannot decay. The supercurrent flows around the torus, in principle, forever.

Since the superfluid is a quantum system, the Goldstone bosons are quantum excitations. Like sound waves in a medium, the individual quantum excitations have energies

$$E(k) = \hbar c k \ , \tag{24}$$

where c is the speed of sound in the fluid, about 200 m/sec in superfluid He^4. These excitations are called *phonons*.

Landau used the idea that, in a superfluid, phonons are the only low-lying excitations, to give a more intutive derivation of the frictionless flow of a superfluid. In a normal fluid, the energy of each particle is

$$\frac{\hbar^2 k^2}{2m} \tag{25}$$

so there are many low-lying excitations and many routes to dissipate the energy of a flow. However, in a superfluid, the only low-lying excitations are the phonons. It is possible to measure the excitation spectrum in He^4 by X-ray or neutron scattering. The result is

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The slope at $k = 0$ is c. The minimum at finite momentum corresponds to an excitation called the "roton", with

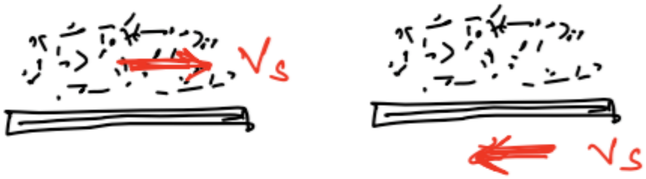
$$E(k) = \Delta + \frac{(k-k_0)^2}{2M} \ . \tag{27}$$

You can think of the roton as a quantized vortex ring, a smoke ring of Helium atoms.

Now consider a superfluid moving across a fixed wall with velocity $\vec{v}_s$. We can analyze the loss of energy by the fluid by going to the fluid's rest frame and then boosting back to the lab by Gallilean transformations. In non-relativistic mechanics, let the energy and momentum of a particle in one frame of reference be $(E, \vec{p})$. Then if the particle is boosted by velocity $\vec{V}$, the energy and momentum become

$$\vec{p}' = \vec{p} + m\vec{V} \ , \qquad E' = E + \vec{V} \cdot \vec{p} \ . \tag{28}$$

Starting from the lab frame, boost to the frame where the fluid is at rest and the wall is in (backwards) motion,



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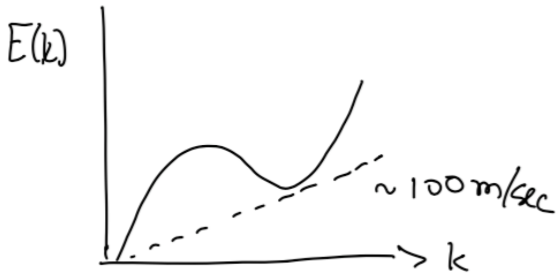
For a normal fluid, the wall can kick a molecule at $p = 0$ to some small but finite momentum. Boosting back to the lab frame, the energy given to the fluid is

$$E_{\text{lab}} = \frac{p^2}{2m} - \vec{v}_s \cdot \vec{p} \tag{30}$$

For small p, this can be negative, so the fluid can lose energy to the wall. However, for a superfluid, the minimum energy excitation with momentum p has energy cp. Then the energy of this excitation in the lab frame is

$$E_{\text{lab}} = cp - \vec{v}_s \cdot \vec{p} \tag{31}$$

This is always positive if $|\vec{v}_s| < c$. Under this condition, the fluid cannot lose energy. For the realistic spectrum of helium,



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superfluid velocities up to about 100 m/sec can flow frictionlessly.

A superfluid with a general supercurrent has a Φ expectation value of the form

$$\langle\Phi\rangle = \sqrt{\rho_s}, e^{i\alpha(x)} \tag{33}$$

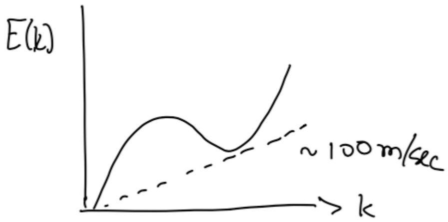
giving the superfluid velocity

$$\vec{v}_s = \vec{\nabla}\alpha(x) \ . \tag{34}$$

This is an irrotational flow,

$$\vec{\nabla} \times \vec{v}_s = 0 \ . \tag{35}$$

Still, it is possible to produce a rotating superfluid. For example, we can spin up a bucket of helium and then let it cool below T_c. This produces a vortex



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of the sort that we discussed at the end of the previous lecture. At the center of the vortex, the superfluid density must vanish by continuity. At a large distance from the center of the vortex, the phase of Φ will rotate followng the angle around the cylinder. But, since Φ must be continuous, the phase must go through 1 or more full rotations. That is, at large distances from the center,

$$\alpha(\phi = 2\pi) - \alpha(\phi = 0) = 2\pi n \ , \tag{37}$$

where n is an integer. This periodicity relation implies that

$$\oint d\vec{x}\cdot\vec{v}_s = \oint d\vec{x}\cdot\vec{\nabla}\alpha = 2\pi n \ . \tag{38}$$

That is, a rotating superfluid has a *quantized vorticity.*

In the case of Bose-Einstein condensation, the system at finite temperature contains both the condensate at $\vec{k} = 0$ and an incoherent gas of atoms at larger momenta. In a superfluid of strongly-coupled atoms, we can describe the state at finite temperature as consisting of the coherent superfluid plus an incoherent gas of phonons and rotons. Then the system has two fluid components, the *superfluid* and the *normal fluid*, which interact weakly with each other. Then the system is described by *two-fluid hydrodynamics.*

The presence of the two fluids was demonstrated in a very illustrative way in 1946 by Elephter Andronikashvili. He build a system with a torsion pendulum consisting of parallel copper disks.

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The frequecy of the pendulum is lower when the pendulum is immersed in a fluid, because the disks drag fluid along with them. He showed that the frequency of the pendulum increases as the temperature is lowered, returning to the value in vacuum as $T \to 0$. In this limit, there is no more normal fluid, and the superfluid is frictionless.