Physics 212 — Problem Set #9

Solutions

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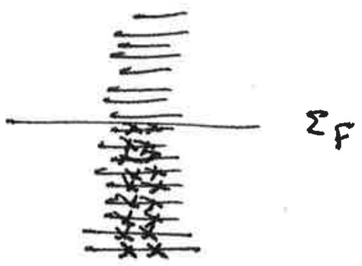
1. a.) In this problem, the spectrum of electron energy levels is continuous, and the levels are filled up to the Fermi energy



In the presence of a single impurity, the energy spectrum is not modified, but the wavefunctions are modified from being simple plane waves. Let an electron energy be

$$E = \varepsilon_F + \epsilon(K) \qquad \epsilon(k) = v_F K$$

For $J=0$, the eigenstate of momentum $\vec{K}$ is a plane wave satisfying

$$\mathcal{H}_0 |\vec{k}\rangle = [\varepsilon_F + \varepsilon(K)] \, |\vec{k}\rangle$$

$$K = |\vec{K}| - \varepsilon_F$$

When J is turned on, these states are mixed together through their interaction with the impurity spin. A basis for the full Hilbert space is

$$|\{\vec{k}, s\}_i; S\rangle$$

where $s = (+,-)$ is the orientation of the electron spin and S is the orientation of the impurity spin. The eigenvalue equation for H is

$$\mathcal{H}|\psi\rangle = (H_0 + JV)|\psi\rangle = \mathcal{E}(k)|\psi\rangle$$

(Since only energy differences will appear, I have dropped the $\mathcal{E}_F$.) Take $|\vec{k}, s; S\rangle$ as a zeroth order eigenstate. Then we can find the full $|\psi\rangle$ by writing

$$(\mathcal{E}(k) - H_0)|\psi\rangle = JV|\psi\rangle$$

and plugging one estimate at order J^n into the right-hand side to get the term in $|\psi\rangle$ at order J^{n+1}. This gives

$$|\psi\rangle = |\vec{k}, s; S\rangle + \dots$$

$$|\psi\rangle = |\vec{k}, s; S\rangle + \frac{1}{\mathcal{E}(k) - H_0} JV|\vec{k}, s, S\rangle + \dots$$

and so on. Then

$$|\psi\rangle = \left(1 + \frac{1}{\mathcal{E}(k) - \mathcal{H}_0} JV + \frac{1}{\mathcal{E}(k) - H_0} JV \frac{1}{\mathcal{E}(k) - H_0} JV + \cdots\right)$$

$$\cdot\, |\vec{k}, s, S\rangle$$

The second term is written explicitly as:

$$\int_{-\mathcal{E}_0/v_F}^{\mathcal{E}_0/v_F} \frac{dK'\, 4\pi k_F}{(2\pi)^3} \frac{|K,s,S\rangle\langle K,s,S|}{\mathcal{E}(K) - \mathcal{E}(K')} J\, A^\dagger_s (\sigma^i)_{ss'} A_{s'} (\Sigma^i)_{SS'} |K', s', S'\rangle$$

I hope it is OK to abbreviate this as

$$\frac{1}{\mathcal{E}(K) - H_0} J\, A^\dagger_s (\sigma^i)_{ss'} A_s (\Sigma^i)_{SS'} |K, s, S\rangle$$

from here on.

b.) I claim that there are pieces of the J^2 term that effectively have this same form. Consider the contribution in which the first V creates an electron at a momentum $\mathbb{K}$ very far above the Fermi surface, and then this electron is annihilated by the second V

$$\underset{\substack{\uparrow \\ \text{creates } K}}{A^\dagger} \sigma^i \underset{\substack{\uparrow \\ \text{annihilates } \mathbb{K}}}{A} \quad \underset{\substack{\nwarrow \\ \text{creates } \mathbb{K}}}{A^\dagger} \sigma^j A \;\leftarrow \text{annihilates } K'$$

Then $A_{\mathbb{K}s} A_{\mathbb{K}'s'} = \{A_{\mathbb{K}s}, A^\dagger_{\mathbb{K}'s'}\} = (2\pi)^3 \delta^{(1)}(\vec{\mathbb{K}} - \vec{\mathbb{K}}')\, \delta_{ss'}$

and the whole structure gives

$$\int \frac{d\mathbb{K}}{(2\pi)^3} 4\pi k_F^2 \frac{A^\dagger_{Ks} (\sigma^i \sigma^j)_{ss'} A_{K's'}}{\mathcal{E}(K) - \mathcal{E}(\mathbb{K})}$$

Finally, since $\varepsilon(K) \gg \varepsilon(k)$ we can approximate

$$\frac{1}{\varepsilon(k) - \varepsilon(K)} \approx \frac{1}{-\varepsilon(K)} = \frac{1}{-v_F K}$$

Then this contribution to the J^2 term is

$$\frac{1}{\varepsilon(k) - H_0} \, J^2 \int^{\varepsilon_0/v_F} dK \, \frac{4\pi k_F}{(2\pi)^3} \, \frac{A^\dagger(\sigma^i \sigma^j) A}{-v_F K} \left(\Sigma^i \Sigma^j\right)_{ss'} |k s' s'\rangle$$

a.) Similarly, consider the contribution in which the annihilation operator in the first term creates a <u>hole</u> in the Fermi sea at a momentum $-K$, and then this hole is filled by the creation operator in the second term.

$$\underset{\substack{\uparrow \\ \text{fills hole at } -K.}}{A^\dagger} \sigma^i \underset{\substack{\uparrow \\ \text{annihilates } K'}}{A} \quad \underset{\substack{\uparrow \\ \text{creates } K}}{A^\dagger} \sigma^j \underset{\nwarrow \text{ creates hole at } -K}{A} \qquad \Sigma^i \Sigma^j$$

To make sense of this, reorder the A operators, with anticommutation,

$$(-1) \underset{\substack{\uparrow \\ \text{creates } K}}{A_s^\dagger} \underset{\nwarrow \text{ annihilates } K'}{A_{s'}} \cdot \underbrace{A^\dagger_{s''} A_{s'''}}_{\text{gives } \delta_{s'' s'''}} (\sigma^i)_{s'' s'} (\sigma^j)_{s s'''}$$

$$= - A^\dagger (\sigma^j \sigma^i)_{ss'} A_{s'} \, (2\pi)^3 \delta^{(3)}(K - K')$$

the hole at $-K$ gives a positive contribution to H_0, equal to $-v_F K$ and this is much larger than $\varepsilon(k)$

Finally, we find

$$(-1)\,\frac{1}{\varepsilon(k)-H_0}\,J^2\int_{-E_0/v_F}\frac{dK\,4\pi k_F}{(2\pi)^3}\,\frac{A^\dagger\sigma^j\sigma^i A}{-v_F|K|}\,(\Sigma^i\Sigma^j)_{ss'}\;|k's',s'\rangle$$

d) Adding the contributions from (b) and (c) we have

$$\frac{1}{\varepsilon(k)-H_0}\,J^2\int_0^{E_0/v_F}\frac{dK\,4\pi k_F}{(2\pi)^3}\,\frac{A^\dagger[\sigma^i,\sigma^j]A}{-v_F K}\,(\Sigma^i\Sigma^j)_{ss'}\;|k's',s'\rangle$$

now $[\sigma^i,\sigma^j] = 2i\,\varepsilon^{ijk}\sigma^k$

and
$$\varepsilon^{ijk}\,\Sigma^i\Sigma^j = \tfrac{1}{2}[\Sigma^i,\Sigma^j]\,\varepsilon^{ijk}$$
$$= \tfrac{1}{2}\left(2i\,\varepsilon^{ijl}\Sigma^l\right)\varepsilon^{ijk}$$
$$= 2i\,\Sigma^k$$

so finally
$$[\sigma^i,\sigma^j]\;\Sigma^i\Sigma^j \;\to\; -4\,\sigma^k\,\Sigma^k$$

and we find
$$\frac{J^2}{\varepsilon(k)-H_0}\int_0^{E_0/v_F}dK\,\frac{4\pi k_F^2}{8\pi^3(-v_F K)}\,(-4)\;A^\dagger\sigma^k A\;\Sigma^k$$

or
$$\frac{J^2}{\varepsilon(k)-H_0}\int_0^{E_0/v_F}\frac{dK}{K}\,\frac{2k_F^2}{v_F\pi^2}\;A^\dagger_s\sigma^k_{ss'}A_{s'}\,\Sigma^k_{ss'}\;|k',s's\rangle$$

This is exactly of the form of the term at the top of p.6, at least for the part of the integral where $\mathbb{K} \gg k$.

e.) Indeed, the expression in part (d) contains

$$\int_0^{E_b/v_F} \frac{d\mathbb{K}}{\mathbb{K}} = \log \frac{E_b/v_F}{0}$$

This is the "Kondo problem".

f.) Integrate out the largest momenta

$$\int_{\frac{1}{\lambda}(E_b/v_F)}^{E_b/v_F} \frac{d\mathbb{K}}{\mathbb{K}} = \log \lambda .$$

The effect of this integration is to convert

$$J \to J + \frac{2k_F^2}{\pi^2 v_F} J^2 \log \lambda$$

As we integrate out successive shells in $\mathbb{K}$, we accumulate <u>positive</u> contributions

g.) If E is the new band cutoff after some integration, $J(E)$ obeys

$$\frac{d}{d\log E} J(E) = - \frac{2k_F^2}{\pi^2 v_F} J^2(E)$$

so that, as E gets <u>smaller</u>, $J(E)$ increases.

the solution of this equation is

$$J(E) = \frac{J_0}{1 - \frac{2k_f^2 J_0}{\pi^2 v_F} \log(E_b/E)}$$

if J_0 is the initial coupling when $E = E_b$

h.) For the ferromagnetic case $J < 0$ the solution is

$$J(E) = - \frac{|J|}{\left(1 + \frac{2k_F}{\pi^2 v_F}|J| \log E_b/E\right)}$$

as $E \to 0$ and we integrate out all of the electrons,

$J(E) \to 0$ and the impurity and the electrons decouple.

i) For the antiferromagnetic case $J > 0$

$$J(E) = + \frac{J}{1 \text{ [illegible] } \frac{2k_F}{\pi^2 v_F} J \log(E_b/E)}$$

and the interaction gets stronger and stronger as we include more electron interactions. Soon we reach a value of J where perturbation theory breaks down. At strong coupling, the impurity spin forms a strongly coupled pair with one electron

$$(\uparrow\downarrow)$$

and removes 1 state from the continuum.