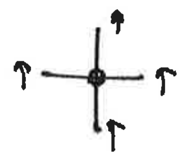
Physics 212 - Problem Set #8

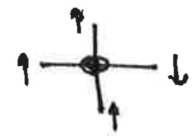
Solutions

1. a.) The 4 distinct spin configurations are

all neighbor spins up
$(\uparrow\uparrow\uparrow\uparrow)$



1 neighbor spin down
$(\uparrow\uparrow\uparrow\downarrow)$



2 neighbor spins down
$(\uparrow\downarrow\uparrow\downarrow\,,\ \uparrow\uparrow\downarrow\downarrow)$

b.)

$\uparrow\uparrow\uparrow\uparrow$ $\quad A^2\,(e^{4J}+e^{-4J})\,e^{4K}$

$\uparrow\uparrow\uparrow\downarrow$ $\quad A^2\,(e^{2J}+e^{-2J})\cdot 1$

$\uparrow\uparrow\downarrow\downarrow$ $\quad A^2\,(2)\cdot 1$

$\uparrow\downarrow\uparrow\downarrow$ $\quad A^2\cdot(2)\;e^{-4K}$

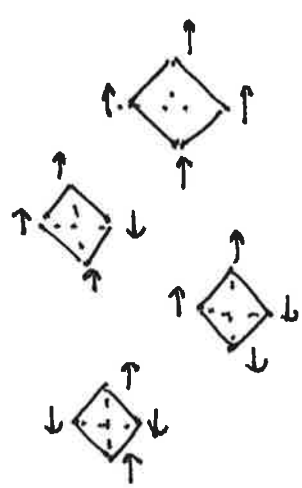
c.)

$\uparrow\uparrow\uparrow\uparrow$ $\quad A'\;e^{4J'}\;e^{2K'}$

$\uparrow\uparrow\uparrow\downarrow$ $\quad A'\,(1)\cdot(1)$

$\uparrow\uparrow\downarrow\downarrow$ $\quad A'\,(1)\;e^{-2K'}$

$\uparrow\downarrow\uparrow\downarrow$ $\quad A'\;e^{-4J'}\cdot e^{2K}$



d)

$$\uparrow\uparrow\uparrow\uparrow \qquad A' e^{4J'} e^{2K'} = A^2 (2\cosh 4J)\, e^{4K}$$

$$\cancel{\uparrow\uparrow\uparrow\downarrow \qquad A' \cdot 1 = A^2 (2\cosh 2J)} \qquad \text{(drop)}$$

$$\uparrow\uparrow\downarrow\downarrow \qquad A' e^{-2K'} = 2A^2$$

$$\uparrow\downarrow\uparrow\downarrow \qquad A' e^{-4J'} e^{2K'} = 2A^2 e^{-4K}$$

e.)

$$① \quad \log A' + 4J' + 2K' = 2\log A + \log(2\cosh 4J) + 4K$$

$$② \quad \log A' - 2K' = 2\log A + \log 2$$

$$③ \quad \log A' - 4J' + 2K' = 2\log A + \log 2 - 4K$$

f.)

$$①-③ \Rightarrow 8J' = \log(\cosh 4J) + 8K$$

$$②-③ \Rightarrow 4J' - 4K' = 4K$$

then

$$J' = \frac{1}{8}\log(\cosh 4J) + K$$

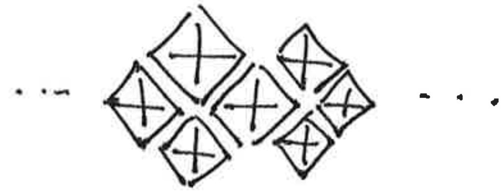
$$K' = J' - K = \frac{1}{8}\log\cosh 4J$$

again

$$J' = \frac{1}{8}\log(\cosh 4J) + K$$

$$K' = \frac{1}{8}\log(\cosh 4J)$$

g.) Assembling the blocks into the new lattice



we have the recursion formula

$$J'' = \frac{1}{4} \log \cosh 4J + 2K$$

$$K'' = \frac{1}{8} \log \cosh 4J$$

h.) The fixed point is given by

$$J_* = \frac{1}{4} \log \cosh 4J_* + 2K_*$$

$$K_* = \frac{1}{8} \log \cosh 4J_*$$

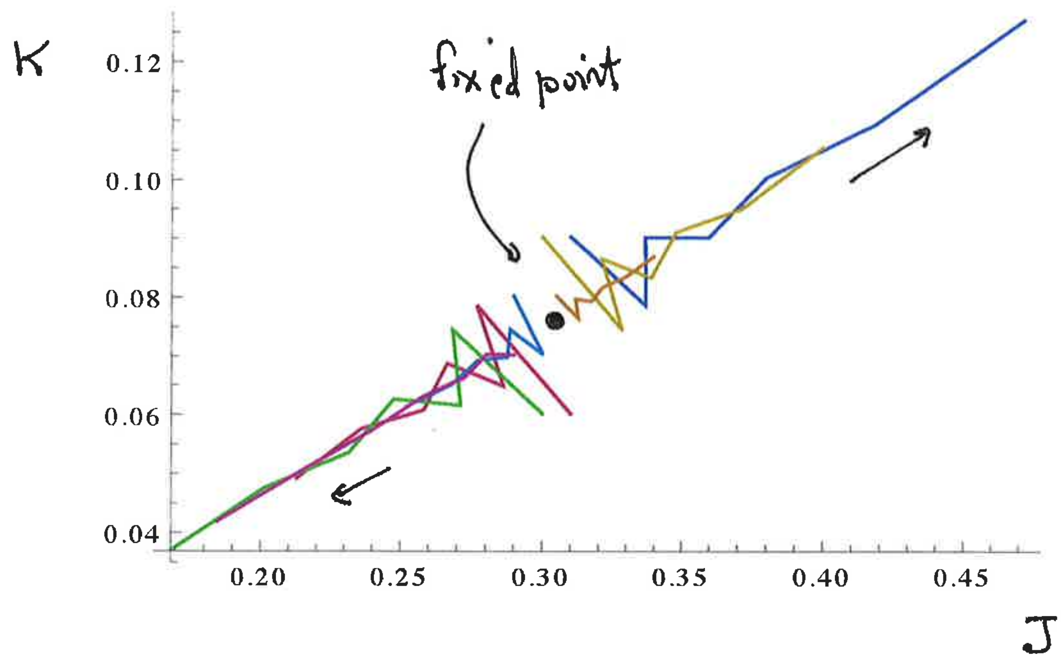
or

$$J_* = \frac{1}{2} \log \cosh 4J_* \,, \quad K_* = \frac{1}{4} J_*$$

the numerical solution is

$$J_* = 0.30469 \qquad K_* = 0.07617$$

i) On the next page, there are some flows generated with Mathematica recursion flows



j) We need

$$\log \cosh 4J = \log \cosh 4J_* + (J - J_*)\left(\frac{4 \sinh 4J_*}{\cosh 4J_*}\right) + \cdots$$

$$= \log \cosh 4J_* + (J - J_*) 4 \tanh 4J_* + \cdots$$

then

$$J'' = \frac{1}{4} \log \cosh 4J_* + (J - J_*) \frac{4}{4} \tanh 4J_*$$

$$+ 2 \frac{1}{8} \log \cosh 4J_* + 2(K - K_*)$$

$$J'' - J_* = \tanh 4J_* \, (J - J_*) + 2(K - K_*)$$

$$K'' - K_* = \frac{1}{2} \tanh 4J_* (J - J_*)$$

or

$$\begin{pmatrix} \Delta J'' \\ \Delta K'' \end{pmatrix} = \begin{bmatrix} \tanh 4J_x & 2 \\ \frac{1}{2} \tanh 4J_x & 0 \end{bmatrix} \begin{pmatrix} \Delta J \\ \Delta K \end{pmatrix}$$

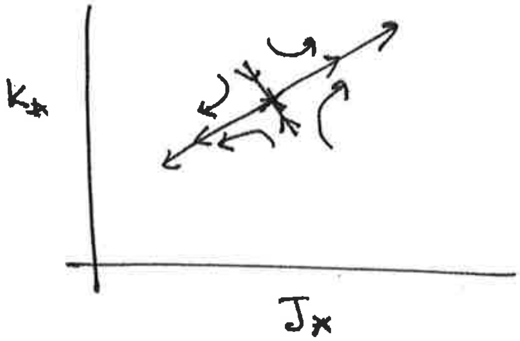
$$\tanh 4J_* \equiv T = 0.8393$$

so $\begin{pmatrix} \Delta J'' \\ \Delta K'' \end{pmatrix} = R \begin{pmatrix} \Delta J \\ \Delta K \end{pmatrix}$ $\quad R = \begin{pmatrix} 0.8393 & 2 \\ 0.41965 & 0 \end{pmatrix}$

from Mathematica:

	λ_1	λ_2
eigenvalues:	1.427	−0.588
(right) eigenvectors:	(0.959, 0.282)	(−0.814, 0.580)

so the flow pattern near the fixed point looks like



k.) If the system starts close to the fixed point, then after n iterations

$$(J - J_*) = \underbrace{(J - J_*)_0}_{\sim \left(\frac{T-T_c}{T_c}\right)} \cdot \lambda_1^n$$

this is order 1 for

$$\log\left(\frac{T-T_c}{T_c}\right) + n \log \lambda_1 \approx 0$$

$$n = \frac{\log\left(\frac{T_c}{T-T_c}\right)}{\log \lambda_1}$$

At the same time, the correlation length goes from ξ_0 to

$$\xi_0 \, 2^{-n/2} \sim \Theta(1)$$

so

$$\xi \sim 2^{n/2} = 2^{\overset{\frac{\log 1/t}{\log \lambda_1}}{\frac{n}{2}}} = e^{-\frac{1}{2}\log 2 \, \log t \, (\log \lambda_1)^{-1}}$$

$$\sim t^{-\frac{1}{2}\log 2 / \log \lambda_1}$$

We can identify $\nu = \frac{1}{2} \frac{\log 2}{\log \lambda_1} = 0.974$

very close to the correct Ising value $\nu = 1$.