Physics 212 - Problem Set #7
Solutions

1.) a.) Begin from

$$G = \int d^3x \left\{ \frac{1}{2m} |\vec{D}\Phi|^2 + \frac{1}{2} a(T-T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right\}$$

where $\vec{D} = \nabla - i \frac{eQ}{c} \vec{A}$

For Φ constant, $\vec{A}=0$, the minimum of G is given by

$$0 = \frac{\partial}{\partial \Phi^*} \left\{ \frac{1}{2} a(T-T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right\}$$

$$= \frac{1}{2} a (T-T_c)\Phi + \frac{b}{2}\Phi|\Phi|^2$$

so $\quad |\Phi|^2 = \frac{a(T-T_c)}{b} \qquad \Phi = \Phi_0 = \left[\frac{a(T_c-T)}{b}\right]^{\frac{1}{2}}$

We can choose a gauge where Φ is real. Then, still with $\vec{A}=0$

$$\Phi(x) = \Phi_0 + \delta\Phi(x)$$

$$G = (\text{const}) + \int d^3x \left\{ \frac{1}{2m} |\nabla\Phi|^2 + \frac{1}{2} a (T-T_c)(\Phi_0 + \delta\Phi)^2 + \frac{b}{4}(\Phi_0+\delta\Phi)^4 \right\}$$

$$= (\text{const}) + \int d^3x \Big\{ \delta\Phi^* \left(-\frac{1}{2m}\nabla^2\right)\delta\Phi + \frac{1}{2} a (T-T_c)\, 2\Phi_0 \delta\Phi + \delta\Phi^2$$
$$+ \frac{b}{4}\left(4\Phi_0^3\delta\Phi + 6\Phi_0^2\delta\Phi^2 + \cdots\right) \Big\}$$

the terms linear in $\delta\Phi$ are

$$(a(T-T_c)\Phi_0 + b\Phi_0^3)\,\delta\Phi = 0 \quad \text{with } \Phi_0 \text{ as above}$$

the quadratic term are

$$G = \int d^3x \left\{ \delta\Phi \left(-\frac{1}{2m}\nabla^2\right)\delta\Phi + \frac{1}{2}a(T-T_c)\,\delta\Phi^2 + \frac{6}{4}b\,\Phi_0^2\,\delta\Phi^2 \right\}$$

$$= \int d^3x \;\delta\Phi \left\{ -\frac{1}{2m}\nabla^2 + \left(\frac{3}{2}b\,\frac{a(T_c-T)}{b}\right) - \frac{1}{2}a(T_c-T) \right\} \delta\Phi$$

$$= \int d^3x \;\frac{1}{2m}\,\delta\Phi \left[-\nabla^2 + 2ma(T_c-T) \right] \delta\Phi$$

The resulting Green's function satisfies.

$$\left[-\nabla^2 + 2ma(T_c-T)\right] G(x,y) = mT\,\delta(x-y)$$

and behaves as

$$G(x,y) \sim e^{-|x-y|/\xi}$$

where

$$\xi = \left[2ma(T_c-T)\right]^{-1/2}$$

b.) Now expand around $\Phi = 0$. The first term are quadratic

$$G = \int d^3x \left[\frac{1}{2m}\Phi^*(-D^2)\Phi - \frac{1}{2}a(T_c-T)\,\Phi^*\Phi \right]$$

$$\vec{D} = \left(\vec{\nabla} - i\frac{eQ}{c}Hx\,\hat{y}\right)$$

$$D^2 = \nabla^2 - 2i\frac{eQ}{c}Hx\,\hat{y}\cdot\vec{\nabla} - \left(\frac{eQ}{c}Hx\right)^2$$

Then

$$\mathcal{O} = -\frac{1}{2m}\left(\nabla^2 - 2i\frac{eQ}{c}Hx\frac{\partial}{\partial y} - \left(\frac{eQ}{c}Hx\right)^2\right) - \frac{1}{2}a(T_c-T)$$

The eigenvalue problem is

$$\mathcal{O}\,\Phi(x) = \lambda\,\Phi(x)$$

c.) Act $\mathcal{O}(H)$ on $\Phi(x,y,z) = e^{ik_y y} e^{ik_z z} f(x)$

$$\mathcal{O}\Phi = \left[-\frac{1}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2i \frac{eQ}{c} H x \frac{\partial}{\partial y} - \left(\frac{eQ}{c} H x\right)^2 + \frac{\partial^2}{\partial z^2} \right) - \tfrac{1}{2} a (T_c - T) \right] \Phi$$

$$= -\frac{1}{2m} \left[\frac{\partial^2}{\partial x^2} - k_y^2 + 2 \frac{eQ}{c} H x\, k_y - \left(\frac{eQ}{c} H x\right)^2 - k_z^2 \right] - \tfrac{1}{2} a (T_c - T) \Big] e^{ik_y y} e^{ik_z z} f(x)$$

so the eigenvalue equation becomes,

$$\left\{ -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2m} \left(\frac{eQ}{c} H x - k_y \right)^2 + \frac{k_z^2}{2m} - a (T_c - T) \right\} f(x) = \lambda f(x)$$

d.)

or

$$\left[-\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2m} \left(\frac{eQ}{c} H \right)^2 (x - x_0)^2 + \frac{k_z^2}{2m} - \tfrac{1}{2} a (T_c - T) \right] f(x) = \lambda f(x)$$

the first two terms can be written as

$$\Omega = \frac{eQH}{mc}$$

$$x_0 = \frac{c}{eQH} k_y$$

$$-\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \tfrac{1}{2} m \Omega^2 (x - x_0)^2$$

which is the harmonic oscillator Hamiltonian. Its eigenvalues are

$$\Omega (n + \tfrac{1}{2})$$

so

$$\lambda = \Omega(n+\tfrac{1}{2}) + \frac{k_z^2}{2m} - \tfrac{1}{2}a(T_c-T)$$

e.) The lowest eigenvalue occurs for $k_z = 0$ and $n=0$

$$\lambda_0 = \frac{1}{2}\Omega - \tfrac{1}{2}a(T_c-T)$$

$$= \frac{1}{2}\frac{eQ}{mc}H - \frac{1}{2}a(T_c-T)$$

f.) This eigenvalue passes through 0, signaling an instability,

at

$$H_* = am(T_c-T)\cdot\frac{c}{eQ}$$

(As long as $\lambda_0 > 0$, the situation $\langle\Phi\rangle = 0$ at that H is stable.)

g.) We found for H_c

$$H_c = \left[\frac{2\pi a^2}{bc}\right]^{1/2}(T_c-T)$$

so

$$\frac{H_*}{H_c} = \frac{am\,(c/eQ)}{\left[\frac{2\pi a^2}{bc}\right]^{1/2}} = \frac{am}{\left[\frac{2\pi a^2}{bc}\left(\frac{eQ}{c}\right)^2\right]^{1/2}}$$

h.) Now $$\xi = \left[2am(T_c-T)\right]^{-1/2}$$

and we found in class: $$\lambda = \left[\frac{4\pi}{mc}\left(\frac{Qe}{c}\right)^2\Phi_0^2\right]^{-1/2}$$

again:

$$\xi = \left[2am(T_c - T)\right]^{-1/2}$$

$$\lambda = \left[\frac{4\pi}{mc}\left(\frac{Qe}{c}\right)^2 \frac{a(T_c - T)}{b}\right]^{-1/2}$$

then

$$\frac{\lambda}{\xi} = \left[\frac{2am}{\frac{4\pi}{mc}\left(\frac{Qe}{c}\right)^2 \frac{a}{b}}\right]^{1/2}$$

$$= \left[\frac{2a^2m^2}{\frac{4\pi a^2}{bc}\left(\frac{Qe}{c}\right)^2}\right]^{1/2} = \frac{am}{\left[\frac{2\pi a^2}{bc}\left(\frac{eQ}{c}\right)^2\right]^{1/2}}$$

so

$$\frac{H_*}{H_c} = \frac{\lambda}{\xi} \quad !$$

If $H_* > H_c$ there is an Abrikosov phase (Type II) and otherwise <u>not</u> (Type I).