Physics 212 — Problem Set #6

Solutions

1. a.) The Landau free energy is

$$G = \int d^2x \left\{ \frac{1}{2}(\vec{\nabla}\Phi^a)^2 + V(\Phi^2) \right\}$$

where $$V(\Phi^2) = \frac{1}{2}a(T-T_c)\Phi^2 + \frac{b}{4}(\Phi^2)^2$$

The value Φ_0 corresponds to the minimum of V. Then

$V(\Phi^2) = V(\Phi_0^2) =$ independent of $\hat{n}$

$$\vec{\nabla}\Phi^a = \Phi_0 \vec{\nabla} n^a$$

so
$$G = \int d^2x \ \frac{1}{2}\Phi_0^2 \sum_i (\vec{\nabla} n^i)^2 + (\text{independent of } \hat{n}(x))$$

b.) let $\hat{n}(x) = \left(\pi^1(x), \pi^2(x), (1-\pi^2)^{\frac{1}{2}}\right)$

$$\vec{\nabla}\hat{n}(x) = \left(\vec{\nabla}\pi^1, \vec{\nabla}\pi^2, -\frac{\pi^{\dot{a}}\vec{\nabla}\pi^{\dot{a}}}{[1-\pi^2]^{1/2}}\right) \qquad \dot{a} = 1,2$$

$$G = \int d^2x \ \Phi_0^2\left[\frac{1}{2}(\vec{\nabla}\pi^{\dot{a}})^2 + \frac{1}{2}\left(\frac{\pi^{\dot{a}}\vec{\nabla}\pi^{\dot{a}}}{1-\pi^2}\right)^2\right]$$

$$= \int d^2x \ \Phi_0^2\left\{\frac{1}{2}(\vec{\nabla}\pi^{\dot{a}})^2 + \frac{1}{2}(\pi^{\dot{a}}\vec{\nabla}\pi^{\dot{a}})^2 + \frac{1}{2}(\pi^{\dot{a}}\vec{\nabla}\pi^{\dot{a}})^2(\pi^{\dot{b}}\pi^{\dot{b}}) + \dots\right\}$$

so there is a unique π^4 term, fixed by the SO(3) rotational symmetry.

c.) Rescale

$$\pi^a = \left(\frac{1}{\beta \Phi_0^2}\right)^{1/2} \Pi^a = \left(\frac{T}{\Phi_0^2}\right)^{1/2} \Pi^a$$

then

$$\beta G = \int d^2x \left\{ \frac{1}{2}(\vec{\nabla}\Pi^a)^2 + \frac{1}{2}\frac{T}{\Phi_0^2}(\Pi^a \vec{\nabla}\Pi^a)^2 \right.$$
$$\left. + \frac{1}{2}(\Pi^a \vec{\nabla}\Pi^a)^2(\Pi^b\Pi^b)\cdot\left(\frac{T}{\Phi_0^2}\right)^2 + \cdots \right\}$$

The nonlinear terms become small — leading to a weak-coupling system — as $T \to 0$ or $T \ll \Phi_0^2$. The coefficient of the Π^{2n} term is $\left(\frac{T}{\Phi_0^2}\right)^{n-1}$.

d.) First, check ~~that~~ the right-hand side of (6) is a unit vector:

$$(\hat{n})^2 = f^2 + g^2 = \left(\frac{2r/a}{1+r^2/a^2}\right)^2 + \left(\frac{1-r^2/a^2}{1+r^2/a^2}\right)^2$$

$$= \frac{(1 - 2r^2/a^2 + r^4/a^4) + 4r^2/a^2}{(1+r^2/a^2)^2}$$

$$= \frac{(1+r^2/a^2)^2}{(1+r^2/a^2)^2} = 1$$

At $\vec{r} = 0$, $\vec{n}(\vec{r}) = (0 \;\; 0 \;\; -1)$ south pole of the sphere

At $\vec{r} \to \infty$, $\vec{n}(\vec{r}) \to (0 \;\; 0 \;\; +1)$ north pole of the sphere

For $\phi = 0$ $\quad \vec{n}(\vec{r}) = (f(r), 0, g(r))$

If we set $\quad g(r) = \cos\chi(r)$, which is fair, because

$g(r)$ is always between 1 and -1, then

$$\chi(r=0) = \pi \qquad \chi(r=a) = \pi/2 \qquad \chi(\infty) = 0$$

since $f^2 + g^2 = 1$, $f = \sin\chi(r)$

then for $\phi = 0$

$$\vec{n}(\vec{r}) = (\sin\chi(r), 0, \cos\chi(r)) \quad \text{line of longitude}$$

and for general ϕ, we find the general line of longitude.

Then $\vec{n}(\vec{r})$ maps the 2-d plane into the sphere in 3-d.

e) $$G = \int d^2x \, \frac{\Phi_0^2}{2} (\vec{\nabla} n^a)^2$$

$$= \int_0^\infty dr \, r \int_0^{2\pi} d\phi \, \frac{\Phi_0^2}{2} \left\{ \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2} f^2 \right\}$$

change variables to $R = r/a$ then

$$dr\, r = a^2 \, dR\, R$$

$$\frac{\partial f}{\partial r} = \frac{1}{a}\frac{\partial f}{\partial R} \qquad \frac{\partial g}{\partial r} = \frac{1}{a}\frac{\partial g}{\partial r} \qquad \frac{1}{r} = \frac{1}{aR}$$

so

$$G = \int_0^\infty dR\, R \int_0^{2\pi} d\phi \, \frac{\Phi_0^2}{2} \left\{ \left(\frac{\partial f}{\partial R}\right)^2 + \left(\frac{\partial g}{\partial R}\right)^2 + \frac{1}{R^2} f^2 \right\}$$

independent of a.

However $\left[(\vec{\nabla}_r n^a)^2\right]^2 = \frac{1}{a^4}\left[(\vec{\nabla}_R n^a)^2\right]^2$

this term scales as $\frac{1}{a^2}$ when added to G

f.) The mapping $\vec{r} \to \hat{n}(\vec{r})$ maps the plane precisely 1 time onto the sphere. Any continuous deformation of this mapping will also have this property. Then it is not possible to continuously deform any of these mappings to the constant mapping $\vec{r} \to \hat{n}_0$ (a constant). They belong to their own topological class. The mapping in this class with minimum G will be a nontrivial topologically stable configuration.

g.) Let's first evaluate T for the mapping from the sphere to the sphere $\vec{n}(\theta,\phi) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$

$$T = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; \frac{\partial n^i}{\partial \theta} \frac{\partial n^j}{\partial \phi} \varepsilon^{ijk} n^k \cdot 2$$

$$\frac{\partial \hat{n}}{\partial \theta} = (\cos\phi \cos\theta, \sin\phi \cos\theta, -\sin\theta)$$

$$\frac{\partial \hat{n}}{\partial \phi} = \sin\theta (-\sin\phi, \cos\phi, 0)$$

$$n = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

The three vectors here are 3 mutually orthogonal unit vectors, so $\frac{\partial \hat{n}}{\partial \theta} \times \frac{\partial \hat{n}}{\partial \phi} \cdot \hat{n} = \sin\theta \cdot \underline{1}$

Then $T = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; 2\sin\theta = 2 \cdot 2 \cdot 2\pi = 8\pi$

If you had chosen to evaluate:

$$T = \int d\theta \sin\theta \, d\phi \left(\frac{\partial n^i}{\partial\theta}\right)\left(\frac{1}{\sin\theta}\frac{\partial n^j}{\partial\phi}\right) \varepsilon^{ijk} n^k$$

you would have found the same answer.

Next, evaluate T for $\hat{n} = (f(r)\cos\phi, f(r)\sin\phi, g(r))$

$$T = \int_0^\infty dr \, r \int_0^{2\pi} d\phi \left(\frac{\partial \hat{n}^i}{\partial r}\right)\left(\frac{1}{r}\frac{\partial n^j}{\partial\phi}\right)\varepsilon_{ijk} n^k \cdot 2$$

$$\frac{\partial \hat{n}^i}{\partial r} = \left(\frac{\partial f}{\partial r}\cos\phi, \frac{\partial f}{\partial r}\sin\phi, \frac{\partial g}{\partial r}\right)$$

now

$$\frac{\partial f}{\partial r} = \frac{2/a\,(1+r^2/a^2) - 2r/a \cdot 2r/a^2}{(1+r^2/a^2)^2} = \frac{2/a\,(1-r^2/a^2)}{(1+r^2/a^2)^2}$$

$$= g \cdot \left(\frac{2/a}{1+r^2/a^2}\right)$$

$$\frac{\partial g}{\partial r} = \frac{-2r/a^2\,(1+r^2/a^2) - (1-r^2/a^2)(2r/a^2)}{(1+r^2/a^2)^2}$$

$$= -\frac{2/a\,(2r/a)}{(1+r^2/a^2)^2} = -\frac{2/a}{1+r^2/a^2}\cdot f$$

so

$$\frac{\partial \hat{n}^i}{\partial r} = \left(\frac{2/a}{1+r^2/a^2}\right)(g\cos\phi, g\sin\phi, -f)$$

$$\frac{1}{r}\frac{\partial n^i}{\partial\phi} = \frac{1}{r}\frac{2/a}{(1+r^2/a^2)}(-f\sin\phi, f\cos\phi, 0)$$

$$n^i = (f\cos\phi, f\sin\phi, g)$$

again, there are 3 orthogonal unit vectors, so

$$\frac{\partial \hat{n}}{\partial r} \times \frac{1}{r}\frac{\partial \hat{n}}{\partial\phi} \cdot \hat{n} = \frac{1}{r}\frac{2/a}{(1+r^2/a^2)} \cdot \frac{2r/a}{1+r^2/a^2} \cdot 1$$

$$T = \int_0^\infty dr\, r \int_0^{2\pi} d\phi \;\frac{4/a^2}{(1 + r^2/a^2)^2} \cdot 2$$

$$= \frac{8\pi}{a^2} \int_0^\infty \frac{1}{2}\, dr^2 \frac{1}{(1+r^2/a^2)^2} \cdot 2$$

$$= 8\pi$$

This corresponds to 1 covering of the sphere.

Finally, note that the structure

$$\int d^2x \; \varepsilon_{ab} \frac{\partial n^i}{\partial x^a} \frac{\partial n^j}{\partial x^b}$$

is independent of the coordinate system. Change variables from x to y

$$\int d^2x = \int d^2y \cdot \left|\det\left(\frac{\partial x^i}{\partial y^j}\right)\right|$$

$$\varepsilon_{ab} \frac{\partial n^i}{\partial x^a} \frac{\partial n^j}{\partial x^b} = \varepsilon_{ab} \frac{\partial y^c}{\partial x^a} \frac{\partial y^d}{\partial x^b} \frac{\partial n^i}{\partial y^c} \frac{\partial n^j}{\partial y^d}$$

$$= \det\left(\frac{\partial y^c}{\partial x^a}\right) \varepsilon_{cd} \frac{\partial n^i}{\partial y^c} \frac{\partial n^j}{\partial y^d}$$

so

$$\int d^2x \; \varepsilon_{ab} \frac{\partial n^i}{\partial x^a} \frac{\partial n^j}{\partial x^b} = \int d^2y \; \varepsilon_{ab} \frac{\partial n^i}{\partial y^a} \frac{\partial n^j}{\partial y^b}$$

only a change of sign is possible, if $\det\left(\frac{\partial x}{\partial y}\right)$ is negative.

so in general

$$T = \int d^2x \; \varepsilon_{ab} \frac{\partial n^i}{\partial x^a} \frac{\partial n^j}{\partial x^b} \varepsilon^{ijk} n^k = \pm 8\pi$$

for any mapping of the plane ~~into~~ the sphere. T is a topological invariant. For a mapping of the plane that covers the sphere n times, $\quad T = 8\pi n$

Each value of n gives its own topological class.