Physics 212 – Problem Set #5

Solutions

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1. (a.) In computing $\mathcal{H}|0\rangle$, all terms are of the form

$$\vec{\sigma}_i \cdot \vec{\sigma}_j \, |(\uparrow)_i (\uparrow)_j\rangle$$

then

$$\left.\begin{array}{l} \sigma_i^1 \sigma_j^1 \, |(\uparrow)_i (\uparrow)_j\rangle = 1 \cdot 1 \cdot |(\downarrow)_i (\downarrow)_j\rangle \\ \sigma_i^2 \sigma_j^2 \, |(\uparrow)_i (\uparrow)_j\rangle = i \cdot i \; |(\downarrow)_i (\downarrow)_j \end{array}\right\} \begin{array}{l} \text{sum} \\ = 0 \end{array}$$

$$\sigma_i^3 \sigma_j^3 \, |(\uparrow)_i (\uparrow)_j\rangle = 1 \cdot 1 \cdot |(\uparrow)_i (\uparrow)_j\rangle$$

so

$$\vec{\sigma}_i \cdot \vec{\sigma}_j |0\rangle = 1 \cdot |0\rangle$$

$$\mathcal{H}|0\rangle = -J \cdot 3N \cdot |0\rangle \quad \text{or} \quad E_0 = -3JN$$

(b.)

$$\sigma^1 = \sigma^+ + \sigma^- \qquad \sigma^2 = -i\sigma^+ + i\sigma^-$$

$$(\sigma_i^1 \sigma_j^1 + \sigma_i^2 \sigma_j^2) = (\sigma_i^+ + \sigma_i^-)(\sigma_j^+ + \sigma_j^-) + (-i\sigma_i^+ + i\sigma_i^-)(-i\sigma_j^+ + i\sigma_j^-)$$

$$= (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-) \cdot (1-1) + (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)(1+1)$$

so

$$(\sigma_i^1 \sigma_j^1 + \sigma_i^2 \sigma_j^2) = 2(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)$$

then

$$\vec{\sigma}_i \cdot \vec{\sigma}_j = \sigma_i^3 \sigma_j^3 + 2(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)$$

(c.) Let $\{k\}$ be the nearest neighbors to j. Then

$$\mathcal{H} = -J\Big[\sum_{m,n \neq k,j}' \vec{\sigma}_m \cdot \vec{\sigma}_n + \sum_k \vec{\sigma}_k \cdot \vec{\sigma}_j\Big]$$

$$\sum_{m,n\neq k,j}' \vec{\sigma}_m \cdot \vec{\sigma}_n \,|(\downarrow)_j\rangle = 1 \cdot |(\downarrow)_j\rangle$$ just as with $|0\rangle$

$$\sum_k' \vec{\sigma}_k \cdot \vec{\sigma}_j \,|(\downarrow)_j\rangle = \sum_k' \vec{\sigma}_k \vec{\sigma}_j \,|(\uparrow)_k (\downarrow)_j\rangle$$

$$= \sum_k \left[\sigma^3_k \sigma^3_j + 2(\sigma^+_k \sigma^-_j + \sigma^-_k \sigma^+_j)\right] |(\uparrow)_k(\downarrow)_j\rangle$$

$$= \sum_k 1\cdot(-1)\,|(\uparrow)_k(\downarrow)_j\rangle + 2\,|(\downarrow)_k(\uparrow)_j\rangle$$

so
$$\mathcal{H}\,|(\downarrow)_j\rangle = \left(-J(3N-6) + J\cdot 6\right)|(\downarrow)_j\rangle + \sum_k (-J\cdot 2)\,|(\downarrow)_k\rangle$$

$$= (E_0 + 12J)\,|(\downarrow)_j\rangle + \sum_k (-2J)\,|(\downarrow)_k\rangle$$

d.)
$$\mathcal{H}\,|\vec{k}\rangle = \sum_j e^{i\vec{k}\cdot\vec{j}}\,|(\downarrow)_j\rangle$$

$$= (E_0 + 12J)\,|\vec{k}\rangle - 2J \sum_j \sum_\nu e^{i\vec{k}\cdot\vec{j}} \left(|(\downarrow)_{j+\nu}\rangle + |(\downarrow)_{j-\nu}\rangle\right)$$

$$= (E_0 + 12J)\,|\vec{k}\rangle - 2J \sum_{j,\nu}\left[e^{ik(j+\nu)} e^{-ik\cdot\nu}\,|(\downarrow)_{j+\nu}\rangle + e^{ik(j-\nu)} e^{+ik\nu}\,|(\downarrow)_{j-\nu}\rangle\right]$$

$$= (E_0 + 12J)\,|\vec{k}\rangle - 2J\,|\vec{k}\rangle \sum_\nu \left(e^{i\vec{k}\cdot\vec{\nu}} + e^{-i\vec{k}\cdot\nu}\right)$$

$$= \left[E_0 + 12J - 4J \sum_\nu (\cos \vec{k}\cdot\vec{\nu})\right] |\vec{k}\rangle$$

so

$$E_k = E_0 + \sum_{a=x,y,z} 4J(1 - \cos k^a)$$

$$\epsilon_k = 4J\left[(1-\cos k^x) + (1-\cos k^y) + (1-\cos k^z)\right]$$

(e) As $k \to 0$ $\quad 1-\cos k \sim \frac{k^2}{2}$ so

$$\epsilon_k \sim 2J|\vec{k}|^2 + O(k^4) \to 0 \text{ as } k \to 0$$

If $\vec{S}$ is the total spin angular momentum and $\vec{S}_i$ is the spin on each site

$$S_i^- |(\uparrow)_i\rangle = \sqrt{2}\, |(\downarrow)_i\rangle$$

Since $\vec{S} = \sum_i \vec{S}_i$,

$$S^- |0\rangle = \sqrt{2} \sum_j |(\downarrow)_j\rangle$$

so at $\vec{k}=0$, $|\vec{k}\rangle$ is a rotation of the original all-spins-up configuration. The problem is symmetric under spin rotations, so

$$E_k \xrightarrow[k\to 0]{} E_0$$

(f) Actually, I showed in (e)

$$\epsilon_k = 2J k^2 + O(k^4)$$