Physics 212 — Problem Set #4
Solutions

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a.) The possible structures are

$$d_x^2 \qquad d_y^2 \qquad d_x d_y$$

Under a $90°$ rotation, these go to $\quad d_x \to d_y \quad d_y \to -d_x$

$$d_x^2 \to d_y^2 \qquad d_y^2 \to d_x^2 \qquad d_x d_y \to -d_x d_y$$

so the invariant is only $\quad d_x^2 + d_y^2$

then

$$G = \int d^3x \left\{ \tfrac{1}{2}(\vec\nabla d_x)^2 + \tfrac{1}{2}(\vec\nabla d_y)^2 + \tfrac{1}{2} A(T)\,(d_x^2 + d_y^2) \right\}$$

b.) Similarly, the only invariants at quartic order are

$$d_x^4 + d_y^4 \qquad d_x^2 d_y^2$$

c.) If $\quad B < 0$, the B term is negative for $d = (d_x^2 + d_y^2)^{1/2} \to \infty$. Along the lines $d_x = 0$ or $d_y = 0$, the C term has no effect, so the potential term in G are

$$\tfrac{1}{2} A(d)\, d^2 - \tfrac{1}{4} |B|\, d^4$$

and this is negative at large d, leading to an instability. For $\;B = 0\;$ the system has neutral stability from the quartic term but can be stabilized by the quadratic term.

If $B > 0$, $C < 0$ we can arrange

$$\frac{1}{4} B (d_x^2 + d_y^2) - \frac{1}{2}|C|\, d_x^2 d_y^2$$

$$= \frac{B}{4}(d_x^2 - d_y^2)^2 + \frac{B}{2} d_x^2 d_y^2 - \frac{1}{2}|C| d_x^2 d_y^2$$

$$= \frac{B}{4}(d_x^2 - d_y^2)^2 + \frac{1}{2}(B - |C|)\, d_x^2 d_y^2$$

so C can be negative, but $|C| > 2B$ gives an instability

d) We need to minimize

$$V = \frac{1}{2} A(T)(d_x^2 + d_y^2) + \frac{1}{4} B (d_x^2 + d_y^2)^2 + \frac{1}{2} C\, d_x^2 d_y^2$$

the minimization equations are

$$0 = \frac{\partial V}{\partial d_x} = d_x \left[A + B(d_x^2 + d_y^2) + C\, d_y^2 \right]$$

$$0 = \frac{\partial V}{\partial d_y} = d_y \left[A + B(d_x^2 + d_y^2) + C\, d_x^2 \right]$$

with $B > 0$ $C > -B$

for $A > 0$, in all cases, the expression in brackets is positive, so the only solution is $d_x = d_y = 0$

for $A < 0$, $C = 0$, the minimum is at

$$A + B(d_x^2 + d_y^2) = 0$$

$$d_x^2 + d_y^2 = |A|/B$$

This is a circle in the (d_x, d_y) plane

for $A < 0$ $C > 0$, the minima will minimize the C term for fixed $d = (d_x^2 + d_y^2)^{1/2}$

$$d_y = 0 \qquad d_x = \pm[|A|/B]^{1/2}$$

or

$$d_x = 0 \qquad d_y = \pm[|A|/B]^{1/2}$$

for $A < 0$ $C < 0$ the minima will maximize the C term for fixed d

$$d_x = \pm d_y = \pm[|A|/(2B - |C|)]^{1/2}$$

e.) $T = T_c$ is the boundary between

$$\langle(d_x, d_y)\rangle = 0 \quad \text{and} \quad \langle(d_x, d_y)\rangle \neq 0$$

f.) From part (d), we see that any point on the circle $d = \text{constant} = (d_x^2 + d_y^2)^{1/2} = [|A|/B]^{1/2}$ is a possible thermodynamic state. Among these (they are all equivalent) choose $d_x = +\left[|A|/B\right]^{1/2}, d_y = 0$

Expand G about this point : $d_x = \bar{d} + \Delta_x \quad d_y = \Delta_y$

$$\tfrac{1}{2}(\vec\nabla d_x)^2 + \tfrac{1}{2}(\vec\nabla d_y)^2 + \tfrac{1}{2}A(d_x^2 + d_y^2) + B(d_x^2+d_y^2)^2$$

$$= \tfrac{1}{2}(\vec\nabla\Delta_x)^2 + \tfrac{1}{2}(\vec\nabla\Delta_y)^2 + \tfrac{1}{2}A\left[(\bar{d}^2 + 2\bar{d}\Delta_x + \Delta_x^2) + \Delta_y^2\right]$$
$$+ \tfrac{1}{4}B(\bar{d}^2 + 2\bar{d}\Delta_x + \Delta_x^2 + \Delta_y^2)^2$$

$$= \tfrac{1}{2}(\vec\nabla\Delta_x)^2 + \tfrac{1}{2}(\vec\nabla\Delta_y)^2 + \tfrac{1}{2}A\bar{d}^2 + B\bar{d}^4$$
$$+ (A\bar{d} + B\bar{d}^3)\Delta_x$$
$$+ \left(\tfrac{1}{2}A + \tfrac{6}{4}B\bar{d}^2\right)\Delta_x^2 + \left(\tfrac{1}{2}A + \tfrac{2}{4}B\bar{d}^2\right)\Delta_y^2 + O(\Delta^3)$$

$\bar{d}$ satisfies $\quad A + B\bar{d}^2 = 0$

so the coefficients of Δ_x and Δ_y^2 both vanish

$$= \tfrac{1}{2}(\vec\nabla\Delta_x)^2 + |A|\Delta_x^2 + \tfrac{1}{2}(\vec\nabla\Delta_y)^2 + O(\Delta^3)$$

$\Rightarrow$ (integrate by parts)

$$G = \int d^3x\left\{\tfrac{1}{2}\Delta_x(-\nabla^2 + 2|A|)\Delta_x + \tfrac{1}{2}\Delta_y(-\nabla^2)\Delta_y\right\}$$

Then Δ_x and Δ_y are uncorrelated :

$$\langle \Delta_x(x)\, \Delta_y(y)\rangle = 0$$

so $\langle d_x(x)\, d_y(y)\rangle = 0$, and from class,

$$\langle d_y(x)\, d_y(y)\rangle = \langle \Delta_y(x)\, \Delta_y(y)\rangle$$

$$= G_0(x,y) = \frac{T}{4\pi}\frac{1}{|x-y|}$$

d_y is a Goldstone boson field.

$$\langle d_x(x)\, d_x(y)\rangle = \langle (d + \Delta_x(x))(d + \Delta_x(y))\rangle$$

$$= d^2 + \langle \Delta_x(x)\, \Delta_y(y)\rangle$$

$$= \frac{|A|}{B} + \frac{T}{4\pi |x-y|} e^{-|2A|^{1/2}|x-y|}$$

g) For $C > 0$, we saw in (d) that there are 4 equivalent solutions

$$d_x = \pm \left[\frac{|A|}{B}\right]^{1/2} \qquad d_y = 0$$

$$d_y = \pm \left|\frac{|A|}{B}\right|^{1/2} \qquad d_x = 0$$

write again $\quad d = \left[\frac{|A|}{B}\right]^{1/2} = \left[\frac{a(T_c - T)}{B}\right]^{1/2}$

Choose $\langle d_x\rangle = d \quad \langle d_y\rangle = 0$ to analyze

We need to expand

$$-\frac{1}{2}|A|(d_x^2 + d_y^2) + \frac{B}{4}(d_x^2 + d_y^2)^2 + \frac{C}{2} d_x^2 d_y^2$$

write $\quad d_x = d + \Delta_x \qquad d_y = \Delta_y$

This gives

$$= -\tfrac{1}{2}|A|\left(d^2 + 2d\Delta_x + \Delta_x^2 + \Delta_y^2\right)$$
$$+ \tfrac{1}{4}B\left(d^2 + 2d\Delta_x + \Delta_x^2 + \Delta_y^2\right)^2 + \tfrac{1}{2}C\left(d+\Delta_x\right)^2\Delta_y^2$$

$$= (\text{const}) - \tfrac{1}{2}|A|\left(2d\Delta_x + \Delta_x^2 + \Delta_y^2\right)$$
$$+ \tfrac{1}{4}B\left(d^4 + 4d^3\Delta_x + 6d^2\Delta_x^2 + 2d^2\Delta_y^2\right)$$
$$+ \tfrac{1}{2}C\,d^2\Delta_y^2 \qquad + O(\Delta^3)$$

$$= (\text{const}) + \Delta_x\left[-d|A| + Bd^3\right] \leftarrow = 0$$
$$+ \Delta_x^2\left[-\tfrac{1}{2}|A| + \tfrac{3}{2}d^2B\right] + \Delta_y^2\left[\underbrace{-\tfrac{1}{2}|A| + \tfrac{1}{2}Bd^2}_{=0} + \tfrac{1}{2}Cd^2\right]$$
$$+ O(\Delta^3)$$

so

$$G = \int d^3x \left\{ \tfrac{1}{2}\Delta_x\left(-\nabla^2 + 2|A|\right)\Delta_x + \tfrac{1}{2}\Delta_y\left(-\nabla^2 + Cd^2\right)\Delta_y \right\}$$

then Δ_x and Δ_y are uncorrelated $\langle d_x(x)\, d_y(y)\rangle = 0$

and
$$\langle d_x(x)\, d_x(y)\rangle = \left[\frac{|A|}{B}\right]^{1/2} + \frac{T}{4\pi}\,\frac{e^{-[2A]^{1/2}|x-y|}}{|x-y|}$$

$$\langle d_y(x)\, d_y(y)\rangle = \frac{T}{4\pi}\,\frac{e^{-[Cd^2]^{1/2}|x-y|}}{|x-y|}$$

The correlation length for d_y is
$$\xi^{-1} = \left[a(T_c - T)\frac{C}{B}\right]^{1/2} \quad \text{and } \xi \to \infty \text{ as } C \to 0$$

h.) For $C < 0$ we saw in (d) that there are 4 equivalent solutions

$$d_x = \pm d_y = \pm \mathbb{d} \qquad \mathbb{d} = \left[\frac{|A|}{2B - |C|} \right]^{1/2}$$

Choose $d_x = d_y = + \mathbb{d}$ to analyze

Let $d_x = \mathbb{d} + \Delta_x \quad d_y = \mathbb{d} + \Delta_y$ and expand G

We need

$$(d_x^2 + d_y^2) = 2\mathbb{d}^2 + 2\mathbb{d}(\Delta_x + \Delta_y) + \Delta_x^2 + \Delta_y^2$$

$$(d_x^2 + d_y^2)^2 = 4\mathbb{d}^4 + 8\mathbb{d}^3(\Delta_x + \Delta_y) + 8\mathbb{d}^2(\Delta_x^2 + \Delta_y^2) + 8\mathbb{d}^2 \Delta_x \Delta_y + O(\Delta^3)$$

$$d_x^2 d_y^2 = (\mathbb{d}^2 + 2\mathbb{d}\Delta_x + \Delta_x^2)(\mathbb{d}^2 + 2\mathbb{d}\Delta_y + \Delta_y^2)$$

$$= \mathbb{d}^4 + 2\mathbb{d}^3(\Delta_x + \Delta_y) + \mathbb{d}^2(\Delta_x^2 + \Delta_y^2) + 4\mathbb{d}^2 \Delta_x \Delta_y + O(\Delta^3)$$

in all

$$-\tfrac{1}{2}|A|(d_x^2 + d_y^2) + B/4\,(d_x^2 + d_y^2)^2 - \frac{|C|}{2} d_x^2 d_y^2$$

$$= (\text{const}) + (\Delta_x + \Delta_y)\left[-|A|\mathbb{d} + 2B\mathbb{d}^3 + C\mathbb{d}^3 \right]$$

$$+ (\Delta_x^2 + \Delta_y^2)\left[-\tfrac{1}{2}|A| + 2B\mathbb{d}^2 - \tfrac{1}{2}|C|\mathbb{d}^2 \right]$$

$$+ \Delta_x \Delta_y \left[2B\mathbb{d}^2 - 2|C|\mathbb{d}^2 \right]$$

coeff of $\Delta_x + \Delta_y$: $\quad d\!\!\!/ \left(-|A| + (2B-|C|)\, d\!\!\!/^2\right) = 0$

coeff of $\Delta_x^2 + \Delta_y^2$: $\quad -\frac{1}{2}|A| + 2B\, d\!\!\!/^2 - \frac{1}{2}|C|\, d\!\!\!/^2$

$$= +\tfrac{1}{2}\left(-|A| + (2B-|C|)\, d\!\!\!/^2 + 2B\, d\!\!\!/^2\right)$$

$$= \tfrac{1}{2} \cdot 2B\, d\!\!\!/^2 = \tfrac{1}{2} \cdot \left(\frac{2B}{2B-|C|}\right)|A|$$

coeff of $\Delta_x \Delta_y$: $\quad (2B - 2|C|)\, d\!\!\!/^2 = \dfrac{2(B-|C|)}{2B-|C|}|A|$

so

$$= (\text{const}) + \tfrac{1}{2}\begin{pmatrix}\Delta_x & \Delta_y\end{pmatrix} M \begin{pmatrix}\Delta_x \\ \Delta_y\end{pmatrix}$$

where $\quad M = \dfrac{|A|}{2B-|C|}\begin{pmatrix} 2B & 2(B-|C|) \\ 2(B-C) & 2B \end{pmatrix}$

This would be simpler if we diagonalize the matrix. Define

$$\Delta_+ = \tfrac{1}{\sqrt{2}}(\Delta_x + \Delta_y) \qquad \Delta_- = \tfrac{1}{\sqrt{2}}(\Delta_x - \Delta_y)$$

$$M\,\Delta_+ = (4B - 2|C|)\frac{|A|}{2B-C}\Delta_+ = 2|A|\;\Delta_+$$

$$M\,\Delta_- = 2|C|\,\frac{|A|}{2B-C}\,\Delta_- = \frac{2|C|}{2B-|C|}|A|\,\Delta_-$$

then

$$G = \int d^3x \left\{ \tfrac{1}{2}\left[(\vec\nabla \Delta_+)^2 + 2|A|\,\Delta_+^2\right] + \tfrac{1}{2}\left[(\vec\nabla \Delta_-)^2 + \frac{2|C|}{2B-|C|}\,\Delta_-^2\right] \right\}$$

so
$$\langle \Delta_+(x)\Delta_-(y)\rangle = 0$$
$$\langle \Delta_+(x)\,\Delta_+(y)\rangle = \frac{T}{4\pi|x-y|}\, e^{-[2|A|]^{\frac{1}{2}}|x-y|}$$
$$\langle \Delta_-(x)\ \Delta_-(y)\rangle = \frac{T}{4\pi|x-y|}\, e^{-\left[\left(\frac{2|C|}{2B-|C|}\right)|A|\right]^{\frac{1}{2}}|x-y|}$$

Let
$$\xi_+ = \frac{1}{|2|A||^{\frac{1}{2}}} = \frac{1}{[2a(T_c-T)]^{\frac{1}{2}}}$$
$$\xi_- = \left[\left(\frac{2B-|C|}{2|C|}\right)\frac{1}{2a(T_c-T)}\right]^{\frac{1}{2}}$$

$$\langle d_x(x)\,d_x(y)\rangle = \left\langle \tfrac{1}{\sqrt{2}}(\Delta_++\Delta_-)(x)\ \tfrac{1}{\sqrt{2}}(\Delta_++\Delta_-)(y)\right\rangle$$
$$= \frac{1}{2}\,\frac{T}{4\pi|x-y|}\left(e^{-|x-y|/\xi_-} + e^{-|x-y|/\xi_+}\right)$$

$$\langle d_x(x)\ d_y(y)\rangle = \left\langle \tfrac{1}{\sqrt{2}}(\Delta_++\Delta_-)(x)\ \tfrac{1}{\sqrt{2}}(\Delta_+-\Delta_-)(y)\right\rangle$$
$$= \frac{1}{2}\,\frac{T}{4\pi|x-y|}\left(e^{-|x-y|/\xi_+} - e^{-|x-y|/\xi_-}\right)$$

$$\langle d_y(x)\ d_y(y)\rangle = \left\langle \tfrac{1}{\sqrt{2}}(\Delta_+-\Delta_-)(x)\ \tfrac{1}{\sqrt{2}}(\Delta_+-\Delta_-)(y)\right\rangle$$
$$= \frac{1}{2}\,\frac{T}{4\pi|x-y|}\left(e^{-|x-y|/\xi_+} + e^{-|x-y|/\xi_-}\right)$$

(2) For $T > T_*$, C is positive. As $T \to T_*$ from above, the correlation length in d_y (not in d_x) $\to \infty$. So, long-range correlations develop.

At $T = T_*$, there is a Goldstone mode, and all points on the circle $d = \left[\frac{|A|}{B}\right]^{1/2}$ become equivalent solutions

For $T < T_*$ C is negative. The thermodynamic state shifts to $\langle d_x, d_y \rangle = \langle \pm d, \pm d \rangle$. As T approaches T_* from below, a Goldstone mode appears as the fluctuation tangent to the circle (Δ_-) and $\xi_- \to \infty$.