Physics 212 – Problem Set #2

Solutions

1.) a.) $$x = n_+ - n_-$$

n_+ = # of forward steps n_- = # of backward steps

$$n_+ + n_- = n$$

so
$$x = n - 2n_- = -n + 2n_+$$

then
$$x + n = 2n_+ = \text{even} \quad \text{and}$$
$$|x| \le n$$

(b.) number of paths with (n_+, n_-) is
$$\binom{n_+ + n_-}{n_+} = \frac{n!}{n_+!\, n_-!}$$

In all, there are 2^n possible paths.

So
$$P(x,n) = \frac{1}{2^n}\binom{n}{n_+}$$

$$\sum_{x=-n}^{n} P(x,n) = \sum_{n_-=0}^{n} \frac{1}{2^n}\binom{n}{n-n_-}$$

$$= \frac{1}{2^n}\left(1 + \frac{n!}{(n-1)!\,1!} + \frac{n!}{(n-2)!\,2!} + \cdots + 1\right)$$

$$= \frac{1}{2^n}(1+1)^n = 1 \checkmark$$

(c) For $n \gg |x|$

$$\binom{n}{(n+x)/2} = \frac{n!}{\left(\frac{n+x}{2}\right)!\left(\frac{n-x}{2}\right)!}$$

$$\cong \exp\left[n\log n - \cancel{n} - \frac{n+x}{2}\log\frac{n+x}{2} + \cancel{\frac{n+x}{2}} - \frac{n-x}{2}\log\frac{n-x}{2} + \cancel{\frac{n-x}{2}} \right]$$

$$= \exp\Big[n\log n - \frac{n}{2}\Big(\log\frac{n}{2} + \cancel{\frac{x}{n}} - \frac{1}{2}\frac{x^2}{n^2} + \cdots\Big)$$
$$- \frac{x}{2}\Big(\cancel{\log\frac{n}{2}} + \frac{x}{n} + \cdots\Big)$$
$$- \frac{n}{2}\Big(\log\frac{n}{2} - \cancel{\frac{x}{n}} - \frac{1}{2}x^2/n^2 + \cdots\Big)$$
$$+ \frac{x}{2}\Big(\cancel{\log\frac{n}{2}} - \frac{x}{n} + \cdots\Big) \Big]$$

$$= \exp\left[-n\log\frac{1}{2} + 2\cdot\frac{x^2}{4n^2} - 2\,\frac{x^2}{2n} + O\left(\frac{x^3}{n^2}\right) \right]$$

$$= (\text{const})\ \exp\left[-x^2/2n \right]$$

Now: $$\int_{-\infty}^{\infty} dx\ e^{-x^2/2n} = \sqrt{2\pi n}$$

so the normalized probability distribution is

$$P(x,n) = \frac{1}{\sqrt{2\pi n}}\exp\left[-x^2/2n\right]$$

also
$$\int_{-\infty}^{\infty} dx \; x^2 e^{-x^2/2n} = \sqrt{2\pi n} \cdot n$$
$$\int_{-\infty}^{\infty} dx \; x^4 e^{-x^2/2n} = \sqrt{2\pi n} \cdot 3n^2$$

so $\langle x^2 \rangle = n \qquad \langle x^4 \rangle = 3n^2$

Note well: $(\langle x^2 \rangle)^{1/2} = \sqrt{n}$!

It is possible to derive the normalization of $P(x,n)$ by using a more accurate approximation to Stirling's formula

$$n! \cong \sqrt{2\pi} \exp\left[(n+\tfrac{1}{2})\log n - n\right]$$

$$\frac{n!}{\left(\frac{n+x}{2}\right)!\left(\frac{n-x}{2}\right)!} = \frac{1}{\sqrt{2\pi}} \exp\Big[n\log n + \tfrac{1}{2}\log n \cancel{- n}$$
$$- \frac{n+x}{2}\log\frac{n+x}{2} - \frac{1}{2}\log\frac{n}{2} + \cancel{\frac{n+x}{2}}$$
$$- \frac{n-x}{2}\log\frac{n-x}{2} - \frac{1}{2}\log\frac{n}{2} + \cancel{\frac{n-x}{2}} + \cdots\Big]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\Big[n\log n + \cancel{\tfrac{1}{2}\log n}$$
$$- \frac{n}{2}\log n - \frac{n}{2}\log\frac{1}{2} - \frac{n}{2}\left(\cancel{\frac{x}{n}} - \frac{1}{2}\frac{x^2}{n^2}\right)$$
$$\cancel{-\tfrac{1}{2}\log n} - \tfrac{1}{2}\log\tfrac{1}{2}$$
$$-\cancel{\frac{x}{2}\log n} - \cancel{\frac{x}{2}\log\tfrac{1}{2}} - \frac{x}{2}\left(\frac{x}{n}\right)$$
$$- \frac{n}{2}\log n - \frac{n}{2}\log\tfrac{1}{2} - \frac{n}{2}\left(\cancel{-\frac{x}{n}} - \tfrac{1}{2}x^2/n^2\right)$$
$$- \tfrac{1}{2}\log n - \tfrac{1}{2}\log\tfrac{1}{2}$$
$$+ \cancel{\frac{x}{2}\log n} + \cancel{\frac{x}{2}\log\tfrac{1}{2}} + \frac{x}{2}\left(-\frac{x}{n}\right)\Big]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\log n - \log\frac{1}{2} - n\log\frac{1}{2} - x^2/2n \right]$$

$$= \frac{1}{\sqrt{2\pi n}} \quad 2 \cdot 2^n \quad \exp\left[- x^2/2n \right]$$

$$P(x,n) = \frac{1}{2^n} \times \text{above} = \frac{2}{\sqrt{2\pi n}} e^{-x^2/2n}$$

but $P(x,n) = 0$ for $(x+n)$ odd. To go to the continuum

$$\sum_x P(x,n) \to \frac{1}{2}\int_{-\infty}^{\infty} dx \; P(x,n)$$

so the ~~contin~~ distribution is

$$P_{\text{cont.}}(x,n) = \frac{1}{\sqrt{2\pi n}} e^{-x^2/2n} \quad \text{satisfying} \quad \int_{-\infty}^{\infty} dx \; P_{\text{cont}}(x,n) = 1$$

d.) with the given $p(\vec{\delta})= \begin{cases} \frac{1}{2d} & \text{for } \delta \text{ a next-neighbor} \\ 0 & \text{otherwise} \end{cases}$

$$\langle 1 \rangle = 1$$

$$\langle \vec{\delta} \rangle = 0$$

$$\langle \delta^a \delta^b \rangle = \frac{1}{d} \delta^{ab}$$

then $\langle \vec{x}(n) \rangle = \langle \sum_{i=1}^{n} \vec{\delta}_i \rangle = 0$

$$\langle |\vec{x}(n)|^2 \rangle = \langle \sum_{i,j} \delta_i^a \delta_j^a \rangle$$

$\vec{\delta}_i$ and $\vec{\delta}_j$ are independent, so $\langle \delta_i^a \delta_j^b \rangle = 0$ for $i \neq j$

$$= \sum_{i=1}^{n} \langle \delta_i^a \delta_i^a \rangle = n \qquad \text{since } |\vec{\delta}_i|^2 = 1$$

Again $(\langle |\vec{x}|^2 \rangle)^{\frac{1}{2}} \propto \sqrt{n}$

(e) If the walker reaches $\vec{x}$ in $(n-1)$ steps, the probability of reaching $\vec{x}+\vec{\delta}$ in n steps is $p(\vec{\delta})$.

Then

$$P(\vec{x},n) = \sum_{\vec{\delta}} p(\vec{\delta})\, P(\vec{x}-\vec{\delta}, n-1)$$

(f.) We need to reduce

$$\prod_a \left(\int_{-\pi}^{\pi} \frac{dk^a}{2\pi} \right) e^{i\vec{k}\cdot\vec{x}} \sum_j e^{-i\vec{k}\cdot\vec{y}_j} f(\vec{y}_j)$$

work on the terms in dimension a

$$\int_{-\pi}^{\pi} \frac{dk^a}{2\pi} \quad e^{ik^a x^a} e^{-ik^a y^a}$$

x^a and y^a are lattice points: $x^a = m \quad y^a = n$

If $m = n$, this integral is 1, otherwise, it is 0

so the above

$$= \left(\prod_a \delta_{x^a, y^a} \right) f(\vec{y}) = f(\vec{x})$$

as required.

(g.) Take the Fourier transform of (8)

$$\sum_{\vec{x}} e^{-ik\cdot x}\left(P(\vec{x}, n+1) = \sum_{\delta} p(\delta)\, P(\vec{x}-\vec{\delta}, n)\right)$$

$$= \tilde{P}(\vec{k}, n+1) = \sum_{\delta} p(\delta) \sum_{x} e^{-i\vec{k}\cdot(\vec{x}-\vec{\delta})} e^{-i\vec{k}\cdot\vec{\delta}}\, P(x-\delta, n)$$

$$= \sum_{\delta} p(\delta)\, e^{-i\vec{k}\cdot\vec{\delta}}\, \tilde{P}(k, n)$$

$$= \sum_{a} \frac{1}{2d}\left(e^{-ik^a} + e^{+ik^a}\right) \cdot \tilde{P}(k, n)$$

$$= \left(\frac{1}{d}\sum_{a} \cos k^a\right) \tilde{P}(k, n)$$

Then

$$\tilde{P}(\vec{k}, n) = [\tilde{p}(k)]^n\, \tilde{P}(\vec{k}, 0)$$

but $P(x, 0) = \delta_{x,0}$ so $\tilde{P}(\vec{k}, 0) = e^{-ik\cdot x}\Big|_{x=0} = 1$

$$\tilde{P}(\vec{k}, n) = [\tilde{p}(k)]^n = \left(\frac{1}{d}\sum_{a} \cos k^a\right)^n$$

Note that $\tilde{p}(k=0) = \frac{1}{d} \cdot d = 1$

(h.) $$P(\vec{x}, n) = \prod_{a} \int_{-\pi}^{\pi} \frac{dk^a}{2\pi}\, e^{i\vec{k}\cdot\vec{x}}\, [\tilde{p}(k)]^n$$

$$\sum_{\substack{x \\ \text{on lattice}}} e^{i\vec{k}\cdot\vec{x}} = \prod_{a}\left(2\pi\delta(k^a)\right)$$

so $\sum_{\vec{x}} P(\vec{x}, n) = [\tilde{p}(k^a)]^n \big|_{k=0} = 1$

then $P(\vec{x}, n)$ is properly normalized.

(i) Analyze $\tilde{P}(\vec{k}, n)$ for $|k| \sim \frac{1}{\sqrt{n}}$. For small $\vec{k}$

$$\tilde{p}(\delta) = \frac{1}{d} \sum_a \left(1 - \frac{(k^a)^2}{2} + \frac{(k^a)^4}{4!} - \dots\right)$$

$$= 1 - \underbrace{\frac{|\vec{k}|^2}{2d}}_{\mathcal{O}(\frac{1}{n})} + \underbrace{\frac{\sum (k^a)^4}{4!\, d}}_{\mathcal{O}(\frac{1}{n^2})} - \dots \quad \text{for } k \sim \frac{1}{\sqrt{n}}$$

$$\cong \exp\left[-\frac{|k|^2}{2d} + \dots\right]$$

then $$[\tilde{p}(\delta)]^n = \exp\left[-\frac{n|k|^2}{2d} + \dots\right]$$

note: this is spherically symmetric!

$$P(x, n) = \int \frac{d^d k}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{x}}\, e^{-n|k|^2/2d}$$

do the integral by completing the square:

$$+\frac{n|k|^2}{2d} - i\vec{k}\cdot\vec{x} = \frac{n}{2d}\left(\vec{k} - i\frac{d}{n}\vec{x}\right)^2 + \frac{d}{2n}x^2$$

$$= \frac{n}{2d}(\vec{k}')^2 + \frac{d}{2n}x^2$$

the Gaussian integral is

$$\int \frac{d^d k}{(2\pi)^d} \, e^{-\frac{n|k|^2}{2d}} = \frac{1}{(2\pi)^d}\left(\frac{2\pi d}{n}\right)^{d/2}$$

Now it is clear that $|k|^2 \sim \frac{1}{n}$

$$= \frac{1}{[2\pi n/d]^{d/2}}$$

so

$$P(x,n) = \frac{1}{[2\pi n/d]^{d/2}} \, e^{-\frac{d}{2n}x^2}$$

for n large $|x| \sim \sqrt{n}$ $\left(\langle (x^a)^2 \rangle = \frac{n}{d} \text{ so } \langle |\vec{x}|^2 \rangle = n \right)$

Again, $\tilde{P}(\vec{k},n)\big|_{\vec{k}=0} = 1$ so P is normalized.

(g) The terms we kept in (i) are spherically symmetric, but the next terms are not. At a higher level of approximation, $\tilde{P}(k,n)$ contains the factor

$$\exp\left[+ \frac{n}{4!\,d}\left(\sum_a (k^a)^4\right)\right]$$

which has only the cubic lattice symmetry. For $k \sim \frac{1}{\sqrt{n}}$, the exponent is $\mathcal{O}\left(n \frac{1}{n^2}\right) \sim \mathcal{O}\left(\frac{1}{n}\right)$ and has diminishing importance as n becomes large (k.) $N = \sum_{n=0}^{\infty} P(\vec{0},n)$

$$= \sum_n \prod_a \int \frac{dk^a}{2\pi} \, e^{i\vec{k}\cdot\vec{x}} \Big|_{\vec{x}=0} [\tilde{p}(k)]^n$$

$$= \prod_a \int_{-\pi}^{\pi} \frac{dk^a}{2\pi} \left[1 + \tilde{p}(k) + (\tilde{p}(k))^2 + \cdots\right]$$

$$N = \prod_a \int_{-\pi}^{\pi} \frac{dk^a}{2\pi} \, \frac{1}{1-\tilde{p}(k)}$$

$$= \prod_a \int \frac{dk^a}{2\pi} \, \frac{d}{\left(d - \sum_a \cos k^a\right)}$$

(l.) For $k \to 0$ $\quad d - \sum_a \cos k^a = d - \sum_a \left(1 - \frac{(k^a)^2}{2} + \cdots\right)$

$$= d - d + \frac{|\vec{k}|^2}{2}$$

In 1-dimension

$$\int_0^{\pi} \frac{dk}{2\pi} \, \frac{1}{|\vec{k}|^2} \quad \text{is divergent}$$

also in 2 dimensions

$$\int_0 \frac{d^2k}{2\pi} \, \frac{1}{k^2} \quad \text{is log divergent}$$

so for $d \leq 2$ $\quad N = \infty$ and the walker always returns to the origin

For $d = 3$ and higher, there is no divergence as $k \to 0$. I guessed that a reasonable approximation to N is

$$N \approx 1 + \frac{1}{d}$$

then $d=3$ $\quad N \cong 1 + \frac{1}{3}$

$d=4$ $\quad N \cong 1 + \frac{1}{4}$

If p is the probability of a return to the origin, then

$$N = 1 + p + p^2 + \cdots$$

$$= \frac{1}{1-p}$$

From the above, we conclude that

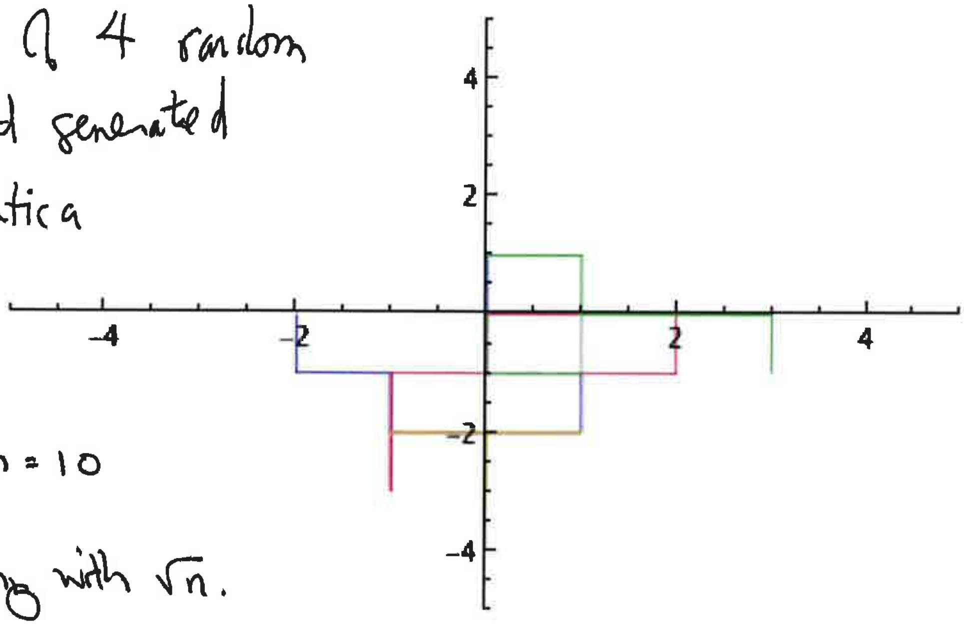
$p \cong \frac{1}{4}$ in $d=3$ $\qquad p \cong \frac{1}{5}$ in $d=4$.

Amazingly G. N. Watson (author of "A Treatise on the Theory of Bessel Functions) was able to find N analytically for $d=3$. See "Polya's Random Walk Constants" in Wolfram MathWorld

Watson: $\quad d = 3 \quad N = 1.516\ldots \quad p = 34.0\%$

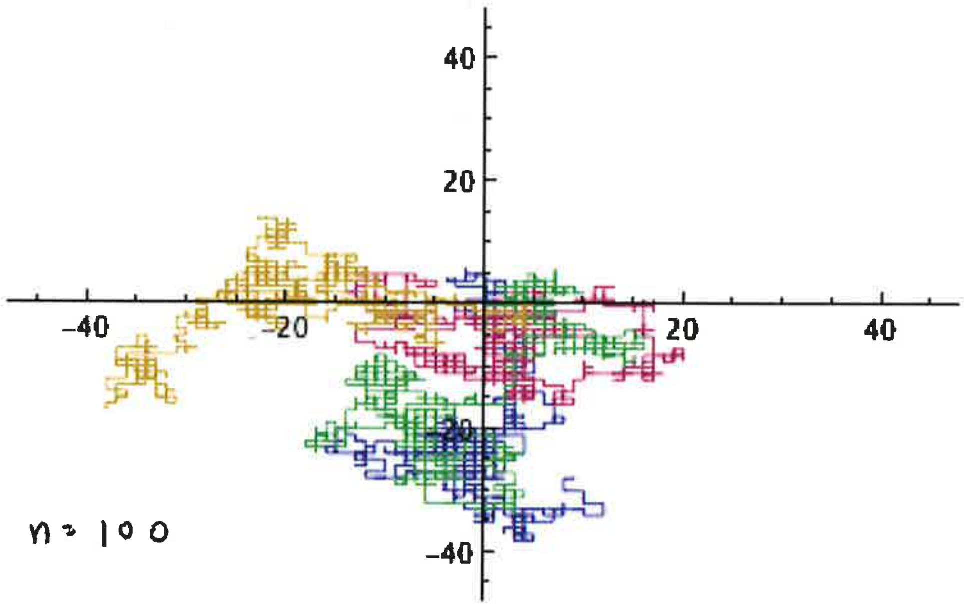
numerically $\quad d = 4 \quad N = 1.24. \quad p = 19.3\%$

Here are sets of 4 random walks in 2-d generated with Mathematica

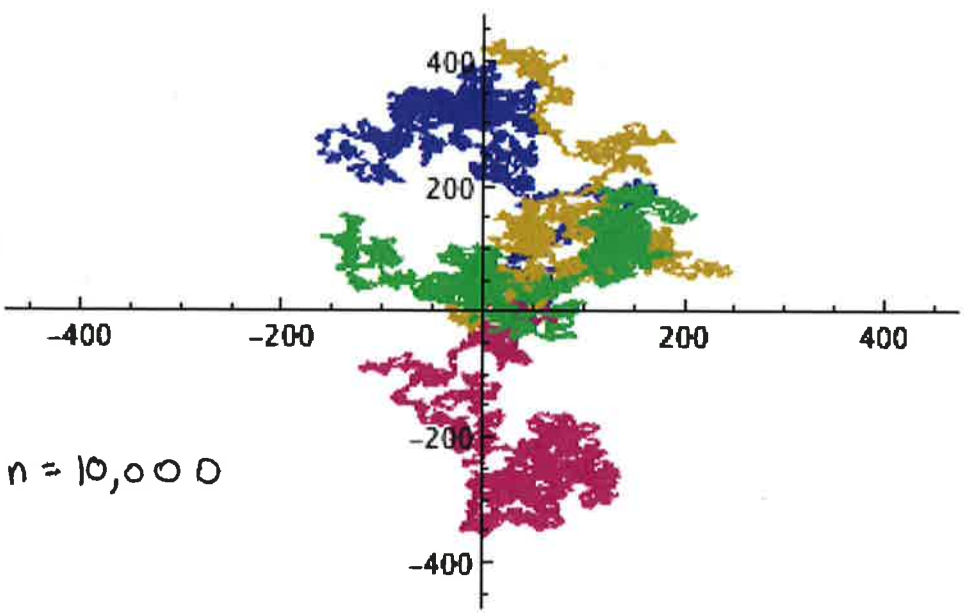


$n = 10$

Note the scaling with $\sqrt{n}$.



$n = 100$



$n = 10,000$