

Physics 212 – Problem Set # 9

(due Friday, December 9)

1. This problem gives a simplified analysis of a system consisting of an isolated magnetic impurity in a metal interacting with conduction electrons. This is the famous “Kondo problem”, introduced by Jun Kondo in 1964. Kondo called attention to a minimum in the electrical resistance at a small but finite temperature. Here we will treat a more basic aspect of this problem, the correction to the interaction strength of the electrons and the impurity, analyzing the effect in the ground state at zero temperature.

The analysis will be based on the model Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{JV} = \int \frac{d^3k}{(2\pi)^3} \epsilon(k) a_{ks}^\dagger a_{ks} + J A_s^\dagger (\sigma^i)_{ss'} A_{s'} \cdot \Sigma^i \quad (1)$$

where \vec{k} is the momentum of an electron, s is the electron spin (+ or –), σ^i is a Pauli sigma matrix acting on the electron, and Σ^i is a Pauli sigma matrix acting on the impurity state. I am assuming that the impurity is a spin-1/2 spin fixed at $\vec{x} = 0$. The operator A_s is the electron quantum field evaluated as $\vec{x} = 0$,

$$A_s = \int \frac{d^3k}{(2\pi)^3} a_{ks} \quad (2)$$

Then A_s annihilates an electron of spin orientation s at $\vec{x} = 0$ and A_s^\dagger creates an electron of spin orientation s at $\vec{x} = 0$. The interaction term can thus change the electron spin orientation and also can change the electron momentum by an arbitrary amount.

For $J = 0$, the ground state of this system is the filled Fermi sea. I will assume that the relevant electrons are in a band from $E = E_{b-}$ to E_{b+} and that (for simplicity) the Fermi energy is in the center of this band. Only energy differences will matter in this problem, so set the zero of energy at the Fermi energy, write

$$\epsilon(k) = v_F \kappa \quad (3)$$

where $\kappa = |k| - k_F$, and let κ run from $-E_b/v_F$ to $+E_b/v_F$. Also approximate

$$d^3k = d\kappa \cdot 4\pi k_F^2 \quad (4)$$

In the ground state for $J = 0$, all states with $\kappa < 0$ are filled and all states with $\kappa > 0$ are empty.

- (a) We would like to solve for the wavefunction of an electron at a momentum \vec{k} just above the Fermi surface. Write the energy of this state as $\epsilon = v_F \kappa$. For $J = 0$, this state is the state of one free electron with, say, spin up

$$|\psi\rangle = |\vec{k}, +\rangle . \quad (5)$$

In the full problem this state obeys the equation

$$\epsilon |\psi\rangle = \mathcal{H} |\psi\rangle \quad (6)$$

Plug in $\mathcal{H} = \mathcal{H}_0 + J\mathcal{V}$ and show that, to second order in J , the solution is

$$|\psi\rangle = \left(\mathbf{1} + \frac{1}{\epsilon - \mathcal{H}_0} J\mathcal{V} + \frac{1}{\epsilon - \mathcal{H}_0} J\mathcal{V} \frac{1}{\epsilon - \mathcal{H}_0} J\mathcal{V} + \dots \right) |\vec{k}, +\rangle \quad (7)$$

- (b) It is possible to write a major contribution to the J^2 term in the same form as the J term. We can add this to the J term and view the sum as an effective electron-spin interaction. To see this, consider the product

$$A_{s_1}^\dagger (\sigma^i)_{s_1 s_2} A_{s_2} \cdot A_{s_3}^\dagger (\sigma^i)_{s_3 s_4} A_{s_4} \quad (8)$$

If $A_{s_3}^\dagger$ creates an electron in an energy level with $\kappa = K > 0$ and A_{s_2} then annihilates that electron, we will have an effective operator in which the electron at the low energy κv_F scatters from the impurity spin in a similar way to the scattering induced by the \mathcal{V} term. Show that this set of intermediate states generates the operator

$$\int_{K>0} \frac{d^3 k}{(2\pi)^3} \frac{1}{(-v_F K)} A_s^\dagger (\sigma^i \sigma^j)_{ss'} A_{s'} \cdot (\Sigma^i \Sigma^j) \quad (9)$$

which can be added to the original interaction term in eq. (1).

- (c) There is another way in which the term of order J^2 can generate a correction to the scattering interaction. If, in eq. (8), A_{s_4} annihilates an electron deep in the Fermi sea, at $\kappa = -K$, this produces a hole that can be filled in by the operator $A_{s_1}^\dagger$. Show that this set of intermediate states produces a contribution to the effective interaction

$$(-1) \cdot \int_{K>0} \frac{d^3 k}{(2\pi)^3} \frac{1}{(-v_F K)} A_s^\dagger (\sigma^j \sigma^i)_{ss'} A_{s'} \cdot (\Sigma^i \Sigma^j) \quad (10)$$

with i and j interchanged and an extra minus sign from fermion operator anti-commutation.

- (d) Add the two contributions from eqs. (9) and (10) and show that the sum gives an operator of precisely the form of the original interaction in eq. (1).

- (e) Since the sum of terms in part (d) gives an effective interaction of the original form, the order J^2 term can be expressed as a modification of J in the original Hamiltonian. Write out the new effective value of J . There is only one problem with this analysis. Show that the final result contains an integral

$$\int_0^{E_b/v_F} \frac{dK}{K} \quad (11)$$

which is divergent as $K \rightarrow 0$.

- (f) Now it is time to invoke a renormalization group analysis. Instead of computing the full integral in (g), compute the integral only over a slice including the largest momenta

$$E_b/\lambda v_F < K < E_b/v_F . \quad (12)$$

with $\lambda > 0$. Fold this contribution back into the Hamiltonian in eq. (1), giving an effective value $J(E)$, where $E = E_b/\lambda$. This is a finite but maybe small shift in J .

- (g) Using the result of (h), write a differential equation for $J(E)$

$$E \frac{d}{dE} J(E) = \dots \quad (13)$$

This is an RG equation.

- (h) Solve the RG equation for the ferromagnetic case $J < 0$ and evaluate the limit of $J(E)$ as $E \rightarrow 0$. What is the conclusion for the physics?
- (i) Solve the RG equation for the antiferromagnetic case $J > 0$. Note that this case has “asymptotic freedom”. What is the conclusion for the physics?