Physics 212 – Problem Set # 8

(due Friday, December 2)

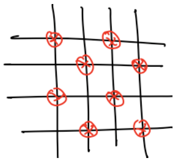
1. Here is another way to work out the lattice renormalization group for the 2-dimensional Ising model on a square lattice:

   I will consider a more general form of the Hamiltonian, with nearest neighbor and next-nearest neighbor interactions

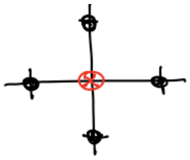
   $$H = J \sum_{\langle ij \rangle} S_i S_j + K \sum_{\langle ik \rangle} S_i S_k - \sum_i \log A \tag{1}$$

   where $\langle ij \rangle$ is a nearest-neighbor pair and $\langle ik \rangle$ is a next-nearest neighbor pair, on opposite sides of an elementary square. The parameter A tracks the overall normalization of the partition function, which will change from step to step in the interation. Set $\beta = 1$; then the inverse temperature is given by some combination of J, K.

   We wish to integrate out spins on odd sites.

   This gives a new square lattice, with a lattice spacing larger by a factor $\sqrt{2}$. To define the transformation, let's concentrate on a block of 5 spins, the odd-site spin that is integrated out, and the 4 neighboring even-site spins.

   We can work out an approximate recursion formula by considering just this cluster of spins, summing over the value of the odd-site spin while keeping the outside, even-site spins fixed. If K is nonzero, longer-range interactions will also be generated. We will close the the system of equations by ignoring these. This will generate an approximate recursion formula that send the parameters (J, K) to a new set of parameters (J'', K'').

(a) Within the block of 5 spins, there are 4 spins that will not be integrated out. For these, there are $2^4 = 16$ possibilities for the spin orientations. However, some of these are related by rotations and by the $S_i \to -S_i$ symmetry. Finally, there are only 4 distinct configurations of the fixed spins. Enumerate these 4 possibilities.

(b) For each, compute the sum of the Boltzmann factors given by the Hamiltonian in eq. (1). To tract the overall normalization, assign the weight A^2 to the block of 5 spins and, after integrating out the odd spin, assign the weight A' to the square with 4 spins. Then, for the case of all even spins up ($\uparrow\uparrow\uparrow\uparrow$), the sum of Boltzmann factors is given by

$$A^2(e^{4J} + e^{-4J})\, e^{4K} \tag{2}$$

(c) If the inner spin is eliminated, the weight for the configuration of 4 spins can be written in terms of new values of the couplings. Giving the new square the overall weight A', we find for the $\uparrow\uparrow\uparrow\uparrow$, the weight

$$A' \exp[4J' + 2K'] \tag{3}$$

corresponding to all nearest neighbor bonds matched along the edges of the square and the two nearest-neighbor bonds across the square also matched. These expressions can be equated to the weights computed in part (b).

(d) Setting the weights equal in the four cases, write 4 equations relating (J', K', A') to (J, K, A). Notice that the equation for the case $\uparrow\uparrow\uparrow\downarrow$ does not involve either J' and K'. I suggest that you drop this equation. In this approximation, we have three equations that give A', J', and K' in terms of A, J, K. Then we do not need to generate any longer-range interactions.

(e) Take the log of each equation. This gives a set of three equations for $\log A'$, J', and K'.

(f) The coefficient A will increase as we integrate out degrees of freedom. However, this is irrelevant for locating the critical point and finding the critical behavior. On the other hand, the interactions J and K can have a fixed point. Algebraically eliminate A and A' and obtain two evolution equations (J', K') in terms of (J, K).

(g) Notice that the nearest-neighbor coupling J' is also generated by a neighboring block of 5 spins. The next-nearest neighbor coupling is not. So set $J'' = 2J'$, $K'' = K'$. This gives the final recursion formula $(J, K) \to (J'', K'')$.

(h) Show that this set of two equations has a fixed point: $(J'', K'') = (J, K)$. Find the location of the fixed point numerically.

(i) Putting in a few values, sketch the recursion as a flow in the (J, K) plane.

(j) Expand the recursion formula to linear order about the fixed point. Let $\Delta J = (J - J_*), \Delta K = (K - K_*)$. The the recursion formula will take the form

$$\begin{pmatrix} \Delta J'' \\ \Delta K'' \end{pmatrix} = \mathbf{R} \begin{pmatrix} \Delta J \\ \Delta K \end{pmatrix} \ . \tag{4}$$

where **R** is (non-symmetric) 2×2 matrix. Diagonalize this matrix, and find the eigenvalues and (right) eigenvectors. Argue that positive eigenvalues of **R** give unstable directions and negative eigenvalues give stable directions. Show that the fixed point has one stable and one unstable direction. This implies that, for systems that are originally in the vicinity of T_c, the effective Hamiltonians on large scales have J and K in a definite relation to one another. The interactions orthogonal to this direction are "irrelevant" in the sense that was described in class.

(k) The value of the exponent ν is given by the rate of instability of the recursion along the unstable direction. Compute the prediction for ν in this approximation scheme.