Physics 212 – Problem Set # 7

(due Friday, November 18)

1. Consider a potentially superconducting material at $T < T_c$ in a large magnetic field H pointing in the $\hat{z}$ direction. As H is decreased, there should be a transition to superconductivity. Let's examine how this can happen. It will be useful to choose the gauge so that $\vec{A} \parallel \hat{y}$ with $\vec{\nabla} \times \vec{A} = H\hat{z}$.

$$\vec{A} = (0, Hx, 0) \tag{1}$$

ith $\vec{D} = (\vec{\nabla} - i(Qe/c)\vec{A})$. Then, as H is decreased, the system should develop an instability at which a $\Phi \neq 0$ turns on.

   (a) The Landau-Ginzburg free energy is

$$G[\Phi] = \int d^3x \left\{ \frac{1}{2m} |\vec{D}\Phi|^2 + \frac{1}{2} a(T - T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right\} \tag{2}$$

   Check that, for zero field, the minimum free energy is at

$$\Phi = \Phi_0 = \left[\frac{a(T_c - T)}{b} \right]^{1/2} \tag{3}$$

   and that the correlation length ξ below T_c is

$$\xi(T) = \Big[2am(T_c - T) \Big]^{-1/2} . \tag{4}$$

   (b) Here we are interested in small fluctuations about $\Phi = 0$, so linearize the Landau-Ginzburg free energy in Φ. Then that expression takes the form

$$G[\Phi] = \int d^3x \left\{ \Phi^* \ \mathcal{O}(H) \ \Phi \right\} \tag{5}$$

   where $\mathcal{O}(H)$ is a second-order differential operator that depends on the parameter H. Write out $\mathcal{O}(H)$ explicitly.

   (c) We now need to diagonalize this operator. Since the problem is uniform in y and z, we can look for eigenfunctions of the form

$$\Phi(x, y, z) = e^{ik_y y} e^{ik_z z} f(x) \tag{6}$$

   Write the eigenvalue equation for $f(x)$.

(d) Show that this equation can be transformed to the equation for a quantum-mechanical harmonic oscillator. Use this observation to find the lowest eigenvalue of $\mathcal{O}(H)$ for large H. Show that this eigenvalue is independent of k_y and occurs for $k_z = 0$.

(e) Show that the the shape of the eigenfunction is a harmonic oscillator wavefunction that is localized in x at a position determined by the value of k_y. By taking superpositions of these wavefunctions, the wavefunction can also be localized in y. These can be the nuclei of flux tubes parallel to $\hat{z}$.

(f) Find the value of $H = H_*$ at which the lowest eigenvalue of $\mathcal{O}(H)$ passes through 0. This is the transition point at which the system with $\Phi = 0$ becomes unstable and superconductivity turns on.

(g) Compare the value of H_* to the value

$$H_c = \left[\frac{2\pi a^2}{bc}\right]^{1/2} (T_c - T) \tag{7}$$

computed in class. If $H_* > H_c$, the system will become superconducting in tubes and will transition continuously to the Abrikosov vortex state. If $H_* < H_c$, the H_c occurs first, giving a first-order transition directly to a state with zero magnetic field inside the superconductor. So the condition $H_* = H_c$ gives the position of the boundary between Type I and Type II superconductivity.

(h) We computed the correlation length ξ in part (a), and we computed the penetration depth λ in class. Relate the condition found in part (g) to the ratio of these two characteristic lengths.