Physics 212 – Problem Set # 6

(due Friday, November 11)

1. Consider a Heisenberg ferromagnet in 2 dimensions. I told you that this system rigorously has no phase transition, but imagine that we have a small enough sample and a low enough temperature that the finite-sized piece of film we are working with has magnetic order. Because the temperature is very low, we can consider the order parameter to have the form

$$\vec{\Phi}(x) = \Phi_0 \hat{n}(x) \tag{1}$$

where $\hat{n}(x)$ is a unit vector in 3 dimensions and all other degrees of freedom in $\vec{\Phi}(x)$ are frozen.

   (a) Plug eq. (1) into the Landau free energy and reduce this to a expression in terms of $\hat{n}(x)$. You should find

$$G[\hat{n}] = \int d^2x \left[\frac{1}{2}\Phi_0^2 \sum_i (\vec{\nabla} n^i)^2 + (\text{const}) \right] \tag{2}$$

   where $i = 1, 2, 3$.

   (b) To get an idea of the physics of this expression, parametrize, for a ground state with $\vec{\Phi} \parallel \hat{3}$,

$$\hat{n}(x) = (\pi^1(x), \pi^2(x), [1 - (\pi^a)^2]^{1/2}) \ , \tag{3}$$

   where $a = 1, 2$ and, here and below, a sum over repeated indices a and i should be understood. Substitute this into (2) and expand to order π^4. Show that this expansion gives a definite prediction for the coefficient of the term quartic in π. This coefficient is fixed by the rotational symmetry of the original problem; changing the quartic coefficient explicitly breaks that symmetry.

   (c) Rescale the variables $\pi^a(x) = C \cdot \Pi^a(x)$ so that, in the expression

$$\exp[-\beta \ G[\hat{n}] \ ] \ , \tag{4}$$

   the derivative terms in the exponent are canonically normalized to

$$\exp\left[-\int d^2x \ \frac{1}{2}(\vec{\nabla}\Pi^a)^2 + \cdots \right] \tag{5}$$

   Find the criteria for the coefficient of the term quartic in Π to be small. Can you estimate the sizes of the coefficients of the Π^6, Π^8, etc., terms?

(d) Consider now the field configuration

$$\hat{n} = (f(r)\cos\phi, f(r)\sin\phi), g(r)) \tag{6}$$

where

$$f(r) = \frac{2r/a}{1 + r^2/a^2} \qquad g(r) = -\frac{1 - r^2/a^2}{1 + r^2/a^2} \tag{7}$$

Show that (6) is indeed a unit vector. Show that this expression satisfies the boundary condition that $\hat{n}(x) \to \hat{3}$ as $r \to \infty$. Show that (6) is a mapping of the 2-dimensional plane onto the unit sphere in 3 dimensions.

(e) Show that, for $G[\hat{n}]$ as we have written it, the configuration (6) gives the same value of $G[\hat{n}]$ for every a. Show, however, if we add higher-derivative terms such as

$$A[(\vec{\nabla} n^i)^2]^2 \tag{8}$$

to $G[\hat{n}]$, the free energy now depends on a and in principle can be stabilized at a given value of a.

(f) Consider deforming the trial solution (6) continuously to the solution that is a local minimum of $G[\hat{n}]$. Argue that this solution is topologically stable. Notice that there is a family of degenerate stable solutions given by rotating the spins around the $\hat{3}$ axis in spin space. These solutions are called "magnetic Skyrmions", after T. H. R. Skyrme. It may be possible to store information on magnetic films in the form of Skyrmions.

(g) Consider the integral

$$T = \int d^2x \; \epsilon_{ab}\epsilon_{ijk}(\nabla_a n^i)(\nabla_b n^j)n^k \tag{9}$$

where $\vec{n}(x)$ is a 3-dimensional unit vector, ϵ_{ijk} is the totally antisymmetric expression with $\epsilon_{123} = 1$, and ϵ_{ab} is the totally antisymmetry expression with $\epsilon_{12} = 1$. Evaluate this integral for a mapping from the unit sphere to the unit sphere

$$\vec{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \tag{10}$$

and $d^2x = d\theta\sin\theta d\phi$. Evaluate T also for the field configuration (6) on the 2-dimensional plane. Show that any 1-to-1 mapping of the plane to the unit sphere with the boundary condition of spin up at infinity gives the same result (up to a sign). What is the significance of T?