Physics 212 – Problem Set # 5

(due Friday, November 4)

1. Consider a Heisenberg ferromagnet with quantum spin-1/2 spins on a 3-dimensional cubic lattice at very low temperature. Let N be the number of sites. The Hamiltonian is

$$\mathcal{H} = -J \sum_{<i,j>} \vec{\sigma}_i \cdot \vec{\sigma}_j \quad (1)$$

where the $\vec{\sigma}$ are the Pauli sigma matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \quad (2)$$

Here i and j are adjacent sites, so the sum is over bonds on the lattice. The matrix $\vec{\sigma}_j$ acts on the spin at the site j. So, the two matrices are not multiplied as matrices; they act on 2-dimensional spinors at different sites. Lattice sites j are vectors $\vec{j}$ with integer components (j_1, j_2, j_3). On a cubic lattice, there are 3 bonds per site; each site attaches to 6 bonds. The ground state of this system has all spins up. Let's try to find a description of the Goldstone bosons.

(a) Show that the state $|0\rangle$ with all spins up is an eigenstate of $\mathcal{H}$ with eigenvalue $E = E_0 = -3JN$.

(b) Show that

$$\sigma_i^1 \sigma_j^1 + \sigma_i^2 \sigma_j^2 = 2(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+) \ , \quad (3)$$

where

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \quad (4)$$

(c) Consider the state $\left|\downarrow_{\vec{j}}\right\rangle$ with all spins up except for one spin down at the site $\vec{j}$. Using eq. (3), compute

$$\mathcal{H} \left|\downarrow_{\vec{j}}\right\rangle \quad (5)$$

(d) Now consider the state

$$\left|\vec{k}\right\rangle = \sum_{\vec{j}} e^{i\vec{k}\cdot\vec{j}} \left|\downarrow_{\vec{j}}\right\rangle \quad (6)$$

Show that this is an eigenstate of the Hamiltonian, and compute the eigenvalue E_k. Write $\epsilon_k = E_k - E_0$. This is the energy of the Goldstone boson of momentum $\vec{k}$.

(e) Show that $\epsilon_k \to 0$ as $\vec{k} \to 0$. Can you give a simple explanation for why this must be so?

(f) Expand E_k to the next term and show that

$$\epsilon_k = Ak^2 + \cdots \tag{7}$$

So in this case the energy of the Goldstone boson vanishes as $\vec{k} \to 0$, as it must, but we do not have $\epsilon_k \sim c|\vec{k}|$ but rather a higher power of k.