Physics 212 – Problem Set # 2

(due Friday, October 14)

1. Many features of statistical mechanics problems appear in the problem of a random walk on a lattice. For example, the diagrams that appear in the high- and low-temperature expansion of the Ising model are very similar to realizations of a random walk. The random walk is much easier to solve than the Ising model, and we can use its exact solution to gain insight. So, in this problem set, we will work out many properties of random walks.

   Let me formalize a random walk as follows: Consider a cubic lattice in d dimensions. Each site has $2d$ nearest neighbors. Consider time n in discrete steps. The walker has position $\vec{x}(t)$, with initial condition $\vec{x}(0) = \vec{0}$. At each time step, the walker moves to one of the nearest neighbors, with equal probability for each choice. Let $P(\vec{x}, n)$ be the probability of the walker being at the lattice point $\vec{x}$ after time step n.

   (a) Consider first the random walk in 1 dimension. Then x is an integer. Argue that $P(x, n)$ is nonzero only if $|x| \leq n$ and only if $(n + x)$ is an even number.

   (b) By counting paths, show that
   $$P(x, n) = \frac{1}{2^n} \binom{n}{(n+x)/2} \tag{1}$$
   It is obvious that we should have
   $$\sum_x P(x, n) = 1 \tag{2}$$
   Prove this from eq. (1).

   (c) Now consider values of n that are very large. Expand the formula derived in (b) using Stirling's formula
   $$n! \sim \exp[n \log n - n + \cdots] \ , \tag{3}$$
   taking x and n now to be continuous variables. Show that $P(x, n)$ becomes a Gaussian distribution when $|x| \ll n$. Adjust the continuum distribution so that it is normalized to
   $$\int_{-\infty}^{\infty} dx \ P(x, n) = 1 \ . \tag{4}$$
   Show that, for this distribution, $\langle x^2 \rangle = n$ (so that indeed $|x| \ll n$) when n is large. Compute $\langle x^4 \rangle$.

   Note: You can derive the correct normalization of $P(x, n)$ by using a slightly more accurate version of Stirling's formula:
   $$n! = \sqrt{2\pi} \exp[(n + \frac{1}{2}) \log n - n + \cdots] \ . \tag{5}$$

(d) Now consider random walks in higher dimensions. Write

$$\vec{x}(n) = \sum_{j=1}^{n} \vec{\delta}_j \ , \tag{6}$$

where each $\vec{\delta}_j$ is drawn from the probability distribtion $p(\vec{\delta})$, where

$$p(\vec{\delta}) = \begin{cases} 1/2d & \vec{\delta} \text{ is a nearest neighbor of } \vec{0} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

By using eq. (5) and the assumption that the $\vec{\delta}_j$ are uncorrelated, show that $\langle \vec{x} \rangle = 0$. Compute $\langle |\vec{x}(n)|^2 \rangle$ and show that this is proportional to n. Thus, the distance of the random walk from the origin grows as $\sqrt{n}$ in any dimension d.

(e) Show that $P(\vec{x}, n)$ obeys the equation

$$P(\vec{x}, n+1) = \sum_{\vec{\delta}} p(\vec{\delta}) \ P(\vec{x} - \vec{\delta}, n) \ . \tag{8}$$

(f) It is not so difficult to solve eq. (7) by Fourier transformation. Here are the appropriate formulae: Let $\vec{y}_j$ be a point on the lattice. Define the Fourier transform of $f(\vec{y}_j)$ as

$$\tilde{f}(\vec{k}) = \sum_j e^{-i\vec{k}\cdot\vec{y}_j} f(\vec{y}_j) \ . \tag{9}$$

where $\vec{y}_j$ is a lattice vector with integer elements. Show that increasing any component of $\vec{k}$ by 2π gives the same result; thus, we can restrict $\vec{k}$ to $-\pi < \vec{k}^a < \pi$ for $a = 1, \ldots, d$. The inverse Fourier transformation is

$$f(\vec{x}) = \prod_a \left(\int_{-\pi}^{\pi} \frac{dk^a}{(2\pi)} \right) \ e^{i\vec{k}\cdot\vec{x}} \ \tilde{f}(k) \tag{10}$$

Verify this formula by plugging (9) into the right-hand side of (10) and showing that it reduces to $f(\vec{x})$. You will need to use the fact that both $\vec{x}$ and $\vec{y}_j$ are points on the lattice.

(g) Now apply (9) to (8) and show that that the Fourier transform of $P(\vec{x}, n+1)$ solves the equation

$$\tilde{P}(\vec{k}, n+1) = \tilde{p}(\vec{k}) \ \tilde{P}(\vec{k}, n) \tag{11}$$

Find the appropriate initial condition, compute $\tilde{p}(\vec{k})$, and solve this equation for $\tilde{P}(\vec{k}, n)$.

(h) Using (10), write the solution for $P(\vec{x}, n)$ as an integral over $\vec{k}$. Show that this probability distribution is normalized.

(i) For $|\vec{x}| \sim \sqrt{n}$, the integral for $P(\vec{x}, n)$ is dominated by the region $|\vec{k}| \sim 1/\sqrt{n} \ll 1$. To analyze this limit, write

$$\tilde{p}(\vec{k}) = \exp[\tilde{q}(\vec{k})] \tag{12}$$

and work out the expansion of $\tilde{q}(\vec{k})$ in powers of $\vec{k}$. Then

$$[\tilde{p}(\vec{k})]^n = \exp[n\tilde{q}(\vec{k})] \tag{13}$$

so we will need terms in $\tilde{q}(\vec{k})$ that are order $1/n$ for $|k| \sim 1/\sqrt{n}$. Since the integral is dominated by small k, we can extend the limits of integration to $(-\infty, \infty)$ without penalty. Now evaluate the integral and find the approximate continuum distribution $P(\vec{x}, n)$. Show that

$$\int_{-\infty}^{\infty} d^d x P(\vec{x}, n) = 1 \tag{14}$$

(j) The underlying lattice is not spherically symmetric, so it is a little surprising that the result of part (i) is a spherically symmetric distribution. Expand $q(\vec{k})$ to the next order and show that there are terms that violate spherical symmetry. And, show that these terms are of relative order $1/n$ as n becomes large.

(k) Going back to the exact formula in part (h), write an integral formula for

$$\mathbf{N} = \sum_{n=0}^{\infty} P(\vec{0}, n) \; . \tag{15}$$

$P(\vec{0}, 0) = 1$, so $\mathbf{N} - 1$ is the average number of times that the random walk passes back through the origin. Show that

$$\mathbf{N} = \prod_a \int_{-\pi}^{\pi} \frac{dk_a}{2\pi} \, \frac{d}{d - \sum_a \cos k_a} \tag{16}$$

(l) Show that (11) is infinite for $d \leq 2$! Then the walker always returns to the origin. The integral is finite for $d > 2$. For $d = 3, 4$, we can crudely extimate the integral by

$$\mathbf{N} = 1 + \frac{1}{d} \tag{17}$$

Estimate the probability for the walker to return to the orgin in these cases. What is the physical interpretation of these results?

If you find these analytical calculations interesting, you might also use your favorite mathematical software to write a computer program to generate random walks in 2 dimensions. For each of $n = 10, 1000, 100,000$, draw 4 representative random walks. Adjust the scale of the plot correctly for each value of n.