Nov. 9

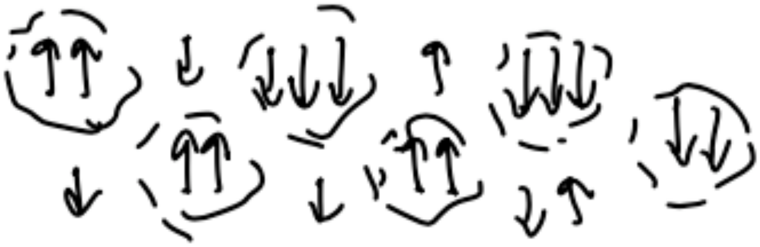
Physics 212 – Statistical Mechanics

Critical Points, Critical Exponents, and Universality

In the remainder of this course, I will concentrate more tightly on the behavior of statistical mechanical systems in the vicinity of a critical point. I have already emphasized that the thermodynamic functions are analytic functions of the parameters of the statistical model except for non-analytic singularities at the critical point. I will now emphasize even more that the physics of these models in the near vicinity of the critical point is especially complex and fascinating.

In this lecture and the following ones, I will describe the physics of the critical point using the language of spins and magnetic order. However, it is important to remember that the same arguments are correct, with appropriate changes of language, for all systems that have a critical point leading to a spontaneously ordered phase.

To begin, think about the pattern of spins in a magnet as T approaches T_c from above. At high temperature, the spin correlation length is small. That is, spins form correlated clusters of size $\xi(T)$, each of which has a random orientation.



(1)

As T decreases, these clusters become larger. In addition, the acquire a nested structure. For $T > T_c$, there is no overall preference for spin up vs. spin down. However, just below $T = T_c$, there is such an overall preference. So, as we approach T_c, we must have clusters of, say, up spins within larger clusters of down spins, within larger clusters, and so forth. Only at the largest length scales does the overall preference become clear.

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Just at T_c, there must be spin fluctuations on all length scales.

This structure of fluctations can be seen directly in the laboratory through the remarkable phenomenon of *critical opalescence*. As a fluid system approaches T_c, the correlation length increases. When $\xi(T)$ becomes of the order of 4000 Å, the fluid will scatter blue light and turn red. This is true even for colorless fluids such as water and CCl_4. When $\xi(T)$ becomes greater than 7000 Å, the fluid scatters light at all visible frequencies and becomes milky. The milkiness dissolves as the liquid and gas phases separate.

What, this means is that a simple, one-scale description of the critical region, as we have in mean field theory and in Landau theory, is bound to be inadequate. We need a better description that can somehow account for the interaction among spins at every length scale.

To approach this theory, I will start by writing a general phenomenological description of the thermodynamic functions and correlation functions in the vicinity of T_c. Let's begin with the spin-spin correlation function. Just at T_c, if there are fluctuations at all length scales, the correlation function should not contain any intrinsic scale, at least at distances $x \gg a$, the atomic spacing, at which a the atom-by-atom structure is smeared out. At these distances, the correlation function should then be a pure power law,

$$\langle S(x)S(y)\rangle = \frac{A}{|x-y|^{d-2+\eta}} \ , \tag{3}$$

or, in Fourier space,

$$\langle S(x)S(y)\rangle = \frac{\tilde{A}}{(k^2)^{1-\eta/2}} \ . \tag{4}$$

Here I have put in the conventional notation for the exponents. In Landau theory, we found forms of this type with, exactly, $T = T_c$, with the behaviors

$$\frac{1}{|x-y|^{d-2}} \quad \text{and} \quad \frac{1}{k^2} \tag{5}$$

Then the parameter η expresses a more general case, with deviations from Landau theory.

Whether or not Landau theory is exactly correct, it is very appealing, based on the above intuition, to assume that the physics at $T = T_c$ is *scale-invariant*. Then any structure that appears at the length scale ℓ is reproduced at the length scale $b\ell$. Then the correlation functions at separations of x and bx should be proportional, with a constant of proportionality that changes systematically with b. This is conveniently written

$$\langle S(x)S(0)\rangle = b^{2D}\,\langle S(bx)S(0)\rangle \tag{6}$$

where I have introduced another new exponent D. A way to interpret this equation is that, within any correlation function.

$$S(x) = b^D S(bx) \tag{7}$$

The parameter D is called the “scaling dimension” of the operator $S(x)$. In Landau theory, $D = (d-2)/2$, In the more general parametrization above, $D = (d-2+\eta)/2$, where η is *a priori* undetermined. In the solved example of the 2-dimensional Ising model at $T = T_c$, the free fermion ψ obeys

$$\langle \psi(x)\psi(y)\rangle \sim \frac{1}{|x-y|} \tag{8}$$

implying $D = 1/2$ or $\eta = 1$ for this operator. I told you also that the spin-spin correlation function turns out to have the form

$$\langle S(x)S(y)\rangle = \frac{A}{|x-y|^{1/4}} \ , \tag{9}$$

implying $D = 1/8$ or $\eta = 1/4$ for the spin operator. Thus, Landau theory is known to be incorrect for the behavior of spin-spin correlation functions in the critical region. It turns out that, by focusing on the values of the scaling dimensions and the exponents in correlation functions, we can find a way to a more exact theory.

Away from $T = T_c$ but still at $x \gg a$, we expect that the only characteristic length in the spin-spin correlation function will be the correlation length. Let

$$t = \left|\frac{T-T_c}{T_c}\right| \tag{10}$$

In Landau theory

$$\xi(t) \sim 1/t^{1/2} \; . \tag{11}$$

In the same spirit as the above, we can allow a more general relation and write

$$\xi(t) \sim 1/t^{\nu} \; , \tag{12}$$

or $\xi(t) = 1/at^{\nu}$, where a is a constant. In the 2-dimensional Ising model, we saw that

$$\nu = 1 \tag{13}$$

since the mass of the fermion depended on T as

$$m_{\psi} \sim |T - T_c| \quad \text{as} \quad T \to T_c \; . \tag{14}$$

This is another failure of Landau theory in this model.

The hypothesis that the spin-spin correlation function has $\xi(T)$ as its only length scale implies

$$\langle S(x)S(0)\rangle = \frac{1}{x^{2D}} f(x/\xi(t)) = \frac{1}{x^{2D}} f(axt^{\nu}) \; . \tag{15}$$

At large x, for T away from T_c ($t \neq 0$),

$$f(xt^{\nu}) \sim e^{-x/\xi(t)} = \exp[-axt^{\nu}] \tag{16}$$

The constant a is inessential, and I will ignore it in the rest of the discussion.

I will now show you that, from the values of η and ν and the assumption that $\xi(t)$ is the only scale in the problem, we can derive the values of the other non-analytic power laws in thermodynamic functions as $T \to T_c$. The powers η and ν and those in thermodynamics functions defined below are called “critical exponents”.

Let’s begin with the spin susceptibility. Above T_c,

$$\chi = \frac{\partial M}{\partial H}\bigg|_T = \int d^d x \;\; \langle S(x)S(0)\rangle \;\; . \tag{17}$$

This quantity is singular as $T \to T_c$; I will write the singularity as

$$\chi \sim t^{-\gamma} \; . \tag{18}$$

To determine γ, first put into the integral formula (17) for χ the scale-invariant result (3) for the correlation function at $T = T_c$. This gives

$$\int d^d x \; \frac{A}{|x|^{d-2+\eta}} \; . \tag{19}$$

The integral is divergent at large x. For $T > T_c$, the integral is cut off at $x = \xi(T)$. Then the value of the integral is given by the above expression with $x = \xi(T)$ as the upper limit. This gives

$$\chi \sim [\,\xi(t)\,]^{2-\eta} \sim t^{-(2-\eta)\nu}\ . \tag{20}$$

We can then identify

$$\gamma = (2-\eta)\nu\ . \tag{21}$$

To evaluate the remaining critical exponents, we will need an additional, slightly stronger, hypothesis called *hyperscaling*. This is an assumption about the scaling of the Gibbs free energy density. Recall that the Gibbs free energy is the integral of a local quantity,

$$G[S] = \int d^dx\ g[S(x)]\ . \tag{22}$$

The density $g[S]$ has the dimensions of $(\text{length})^{-d}$. I will now make the assumption that the contributions to $g[S]$ are dominated by the fluctuations of the spin field at size scale much greater than the original crystal lattice spacing. This is not necessary, but it is highly plausible. At these larger distances, the only relevant length is $\xi(t)$. Then we estimate

$$g[S(x)] \sim [\xi(t)]^{-d} \sim t^{d\nu}\ . \tag{23}$$

Under a change of scale $x \to bx$,

$$\xi(t) \to \xi(t)/b\ , \quad \text{so that} \quad t \to b^{1/\nu} t\ . \tag{24}$$

Also, from (7),

$$S \to b^{(d-2+\eta)/2} S \tag{25}$$

The overall magnetization M is an integral of S and has the same scaling. Then the Gibbs free energy, which is a function of M and T, must have the form

$$G \sim t^{d\nu}\ h(M t^{-(d-2+\eta)\nu/2})\ , \tag{26}$$

where $h(z)$ is an unknown function, but its argument z is constructed to be scale-invariant.

We can determine the thermodynamic scaling laws from this expression. The specific heat is a derivative of this function taken at $M = 0$,

$$C = \frac{\partial^2}{\partial t^2} G\Big|_{M=0} \sim t^{d\nu-2} \tag{27}$$

If we define the exponent α by

$$C \sim t^{-\alpha} \tag{28}$$

then

$$\alpha = 2 - d\nu\ . \tag{29}$$

For $T > T_c$

$$\left.\frac{\partial G}{\partial M}\right|_T = H \ . \tag{30}$$

The magnetic susceptibility is

$$\chi = \left.\frac{\partial M}{\partial H}\right|_T = \left[\left.\frac{\partial^2 G}{\partial M^2}\right|_T\right]^{-1} \ . \tag{31}$$

Differentiating and then setting $M = 0$, we find

$$\chi \sim \left[t^{d\nu} t^{-2\cdot(d-2+\eta)\nu/2}\right]^{-1} = t^{(2-\eta)\nu} \tag{32}$$

This gives the same expression for the exponent γ that we found earlier.

Rearranging the functional dependence of $G(M,T)$ as

$$G = M^{2d\nu/(d-2+\eta)\nu} \ k(tM^{-2/(d-2+\eta)\nu}) \ , \tag{33}$$

or

$$G = M^{2d/(d-2+\eta)} \ k(tM^{-2/(d-2+\eta)\nu}) \ , \tag{34}$$

we can now set $t = 0$ and then differentiate with respect to M to find

$$H = \left.\frac{\partial G}{\partial M}\right|_T \sim M^{(2d/(d-2+\eta))-1} = M^{(d+2-\eta)/(d-2+\eta)} \ . \tag{35}$$

Defining the relation of M and H at $T = T_c$ as

$$M \sim H^{1/\delta} \tag{36}$$

we find

$$\delta = \frac{d+2-\eta}{d-2+\eta} \ . \tag{37}$$

Finally, the minimum of the function $k(x)$ will occur at a fixed value of the argument

$$x_{min} = tM^{-2/(d-2+\eta)\nu} \ . \tag{38}$$

This implies that, if we define the exponent β by the relation

$$M \sim |t|^{\beta} \tag{39}$$

then

$$\beta = \frac{(d-2+\eta)}{2}\nu \ . \tag{40}$$

Let's make a table of these exponents, giving their definitions and their values in the scaling theory and in Landau theory:

Exponent	definition	at	scaling	Landau
η	$\langle S(x)S(0)\rangle \sim 1/x^{d-2+\eta}$	$T = T_c$	—	0
ν	$\xi \sim t^{-\nu}$	$T \to T_c$	—	1/2
α	$C \sim t^{-\alpha}$	$T \to T_c$	$2 - d\nu$	0
β	$M \sim t^{\beta}$	$T \to T_c$	$(d-2+\eta)\nu/2$	1/2
γ	$\chi \sim t^{-\gamma}$	$T \to T_c$	$(2-\eta)\nu$	1
δ	$M \sim H^{1/\delta}$	$T = T_c$	$(d+2-\eta)/d-2+\eta)$	3

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The Landau theory results are predictions for all d. We observe that the d-independent relation $\gamma = (2-\eta)\nu$ is satisfied. Rather oddly, the d-dependent relations are satisfied only in $d = 4$. I will argue, later in the course, that this value is an *upper critical dimension* for these magnetic models.

We have already seen that the Landau results are incorrect for the 2-dimensional Ising model. What about for real 3-dimensional systems? The critical exponents have been measured carefully in a large number of 3-dimensional systems with critical points. In some cases, the exponents have experimental errors in the third signficant figure; you should contemplate the degrees of care and control that such measurements required. Here are some of the results, taken from Zinn-Justin's book *Quantum Field Theory and Critical Phenomena.* I tabulate separately results for systems for which the correponding Landau theory has $n = 1$, 2, or 3. The numbers in parentheses are the experimental errors in the last place quoted.

For $n = 1$ systems:

exponent	value	system
η	0.016(2)	binary liquid
	0.04(2)	β-brass
ν	0.625(5)	binary liquid
	0.65(2)	β-brass
α	0.113(5)	binary liquid
	0.12(2)	liquid-gas
β	0.325(5)	binary liquid
	0.34(1)	liquid-gas
γ	1.240(7)	binary liquid
	1.22(3)	liquid-gas
	1.24 (2)	β-brass

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For $n = 2$ systems:

exponent	value	system
ν	0.672(1)	superfluid He^4
α	-0.013(3)	superfluid He^4

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For $n = 3$ systems:

exponent	value	system
ν	0.70(2)	EuO, EuS
	0.724(8)	$RbMnF_3$
α	-0.011(2)	Ni
β	0.37(2)	EuO, EuS
	0.348(5)	Ni
	0.316(8)	$RbMnF_3$
γ	1.40(3)	EuO, EuS
	1.33(3)	Ni
	1.40(3)	$RbMnF_3$

(44)

Notice that η is generally close to 0, so we can test the scaling relations even for the cases where η is not measured. They are satisfied to a satisfactory degree. For the case of superfluids, the exponents η and β cannot be measured, because they refer to the magnitude of the condensate wavefunction, which is not experimentally observable. The kinetic term of phonons has the form

$$G = \int d^3x \ \frac{1}{2m}\rho_s(T) \ (\vec{\nabla}\phi)^2 \tag{45}$$

where ϕ is the phase of the superfluid order parameter. The corresponding expression for the supercurrent is

$$\vec{J} = \rho(T) \ \vec{\nabla}\phi \tag{46}$$

Naively, $\rho(T)$ is proportional to $\Phi_0^2(T)$. However, it is more rigorous to evaluate $\rho(T)$ by using hyperscaling. This quantity has the dimensions $(\text{length})^{-(d-2)}$. Hyperscaling then implies

$$\rho(T) \sim (\ \xi(T)\ )^{-(d-2)} \sim t^{\nu(d-2)} \ . \tag{47}$$

The measured behavior of $\rho(T)$ near $T = T_c$ is

$$\rho(T) \sim t^{\zeta} \qquad \zeta = 0.67 \tag{48}$$

which is in good agreement with this prediction.

The exponents that I have tabulated above have two more remarkable features:

- The measured exponents have some qualitative similarly to the predictions of Landau theory, but, quantitatively, they deviate significantly.

- However, systems with the same Landau theory have the same values of the critical exponents.

A set of systems with the same values of the critical exponents are called a "universality class". The systems tabulated above form three distinct universality classes.

It is worth emphasizing how unexpected it is to find deviations from the scaling predictions of Landau theory. Of course, Landau theory is only an *ad hoc* phenomenology. But the scaling laws that it predicts follow from straightforward dimensional analysis. In Landau theory, we write

$$\beta G[S] = \int d^d x \; (\beta g[S]) = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla} S)^2 + \frac{1}{2} a(T - T_c) S^2 + \cdots \right\} \tag{49}$$

Notice that I have rescaled S to absorb β, so that the derivative term has a simple, canonical normalization. Then, dimensional analysis requires $(\beta g[S])$ to have the units of $(\text{length})^{-d}$. Similarly, dimensional analysis gives the units of the elements of the integrand,

$$\vec{\nabla} S \sim (\text{length})^{-d/2} \qquad S \sim (\text{length})^{-(d-2)/2} \tag{50}$$

and

$$a(T - T_c) = \xi^{-2} \sim (\text{length})^{-2} \tag{51}$$

If $a(T - T_c)$ provides the only length in the problem, we must find

$$\xi \sim (T - T_c)^{-1/2} \tag{52}$$

and the scaling of the spin-spin correlation function must be such that

$$D = (d - 2)/2 \; . \tag{53}$$

hese simple observations lead to the predictions

$$\eta = 0 \qquad \nu = 1/2 \tag{54}$$

But these predictions turn out not to be correct for real systems.

In a scale-invariant system, the failure of dimensional analysis requires that the relationship of operators at two different scales given by simple dimensional analysis is incorrect for *every* relation of scales x to bx. How can this occur? To solve this problem, we need to look into how we calculate the effect of a scale transformation. We will see, in particular, that the simple scaling predicted by dimensional analysis can be altered by the effects of nonlinear interactions.

In the next three lectures, I will introduce you to a method for analyzing this problem which goes under the unfortunate name of the Renormalization Group. (The name is the responsiblity of Murray Gell-Mann.) The study of the Renormalization Group will lead us to a general picture of the origin of the universality classes.