Nov. 30

Physics 212 – Statistical Mechanics

The Renormalization Group and Critical Exponents - 2

In the previous lecture, I explained the continuum RG approach to the understanding of the critical exponents using the example of models in the universality class of the Ising model. Now I would like to continue that discussion along several directions. In particular, I will address three questions. First, how does this analysis work for Landau theories with higher symmetry? Do we obtain the same predictions for the critical exponents, or different ones? What are the universality classes? Second, can we generate improved predictions by computing more terms the power series in ϵ or, more generally, more terms in the Feynman diagram series? Third, now that we have understood the upper critical dimensionality at $d = 4$ from the RG picture, can we also understand the lower critical dimensionality? I will consider these questions in turn.

We can address the first question directly by using the methods of the previous lecture to analyze the $n > 1$ Landau theories, with spin fields $S^a(x)$, $a = 1, \ldots, n$ and statistical weight

$$\int d^dx \Big\{ \frac{1}{2}(\vec{\nabla} S^a)^2 + \frac{1}{2} m^2 (S^a)^2 + \frac{b}{4}((S^a)^2)^2 \Big\} \tag{1}$$

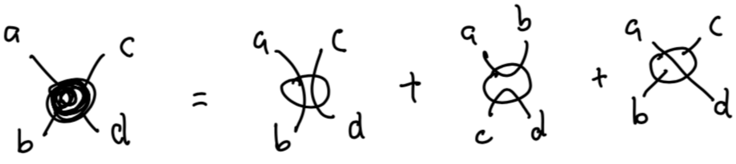
with sums over repeated indices a understood. The Feynman diagrams accounting the interactions are the same, except that we now need to keep track of the a indices. To visualize the index structure of the S^4 vertex, it is useful to write it as

$$\frac{b}{4}\; S^a S^a\, S^b S^b \tag{2}$$

Contractions into this vertex can contract external fields S^a to fields with coupled or distinct a indices. Notice that there are still 4! ways to contract external fields with the four S^a's. But these 4! contractions divide into three distinct classes. For example, if two external fields contract into the first pair of S's and the other two external field contract into the second pair of S's, the indices in each pair of fields are set equal,

$$= \; -2b\, \delta^{ab} \delta^{cd} \tag{3}$$

There are $4 \cdot 2$ contractions with this property, and so I have given this term the overall coefficient $4 \cdot 2 \cdot (-b/4) = -2b$. Other contractions give the other two possible index structures. In all, the 4-point vertex in this model is given by



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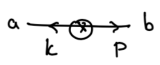
An illustrative example is the calculation of the scaling dimension of the S^2 operator that we studied at the end of the previous lecture. In the model with general n,

$$S^2 = S^c S^c \ . \tag{5}$$

The leading order matrix element of this operator is

$$\left\langle S^a(x)\, S^b(y)\, S^2(z) \right\rangle =$$



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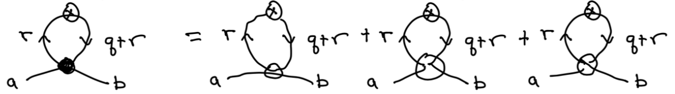
for which the value is

$$\left\langle S^a(x) S^b(y) S^2(z) \right\rangle = \mathrm{FT}\left\{ \frac{1}{k^2}\ 2\delta^{ab}\ \frac{1}{p^2} \right\} , \tag{7}$$

or, in the abbreviated form

$$2\delta^{ab} \tag{8}$$

In the next order, the quartic vertex leads to the diagram



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Notice that the second and third diagrams are equal. All three diagrams contain the same integral over momenta that we studied in the previous lecture. The value of the diagram is then

$$2 \cdot \frac{1}{2} \cdot (-2b) \left[\delta^{ab}\delta^{cc} + 2\delta^{ac}\delta^{cb} \right] \int \frac{d^d r}{(2\pi)^d} \frac{1}{r^2 (q+r)^2} \tag{10}$$

Note that there is still a symmetry factor 1/2. In the factor δ^{cc}, we are instructed to sum the index c over all n values. Then the value of the diagram is

$$\begin{aligned} &2 \cdot \left((-b)[n\delta^{ab} + 2\delta^{ab}] \frac{1}{8\pi} \log \lambda \right) \\ &= 2 \cdot \left(-\frac{(n+2)b}{8\pi^2}\ \log \lambda\ \delta^{ab} \right) \end{aligned} \tag{11}$$

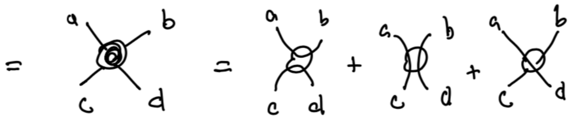
Comparing this expression to (8), we find that the evoiution equation for the coefficient of the S^2 operator is

$$\ell \frac{d}{d\ell}\; m^2 = (2 - \frac{(n+2)b}{8\pi^2})\; m^2 \tag{12}$$

which agrees with our previous result for $n = 1$.

In a similar way, we can compute the new term in the RG equation for the b coeficient. The indices proliferate, and so we will need to track them carefully. The leading-order result for the 4-spin correlation function is now

$$\langle S^a S^b S^c S^d \rangle =$$



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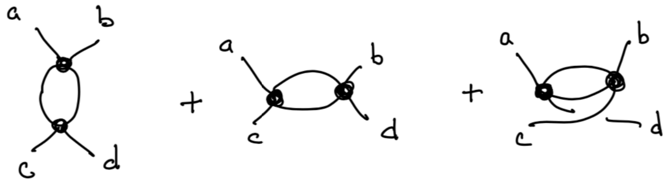
The value of this diagram is

$$\Big\langle S^a S^b S^c S^d \Big\rangle = -2b\Big[\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\Big]\ . \tag{14}$$

The $\mathcal{O}(b^2)$ correction to this result comes from the same three diagrams as before,



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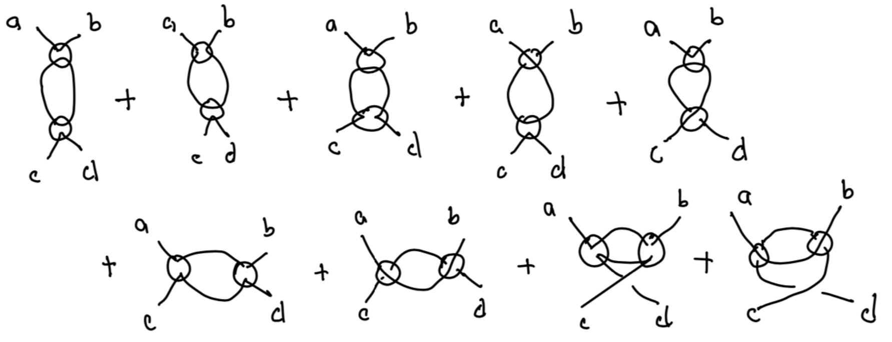
but now we must use the more complicated expression (4) for the vertex. This leads to $3 \cdot 3 \cdot 3 = 27$ terms. To make this simpler, let's concentrate on the 9 terms that lead to the structure $\delta^{ab}\delta^{cd}$. These are



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All of these diagrams have the form of a factor containing the indices times the same integral, which in 4 dimensions is equal to

$$\frac{1}{8\pi^2} \log \lambda\ . \tag{17}$$

The diagrams have a symmetry factor 1/2 as before. Notice that the first diagram (only) has an index loop that is not connected to the external fields. Then part depending on the indices is then

$$\begin{aligned}&\frac{1}{2}\cdot(-2b)^2\ [\delta^{ab}\delta^{cd}\delta^{ee} + 8\ \delta^{ab}\delta^{cd}]\\ &\quad = 2b^2(n+8)\end{aligned} \tag{18}$$

We then find for the RG equation for $b(\ell)$ in 4 dimensions

$$\ell\frac{d}{d\ell}\ b(\ell) = -\frac{(n+8)}{8\pi^2}b^2 \tag{19}$$

In $d = (4-\epsilon)$ dimensions, the RG equation for $b(\ell)$ becomes

$$\ell\frac{d}{d\ell}\ b(\ell) = +\epsilon\ b - \frac{(n+8)}{8\pi^2}b^2 \tag{20}$$

The fixed point of the RG flow is now at

$$b_* = \frac{8\pi^2}{n+8}\epsilon\ . \tag{21}$$

The RG equation for m^2 (12) with the fixed-point value of b is

$$\ell\frac{d}{d\ell}\ m^2 = (2 - \frac{n+2}{n+8})\ m^2\ . \tag{22}$$

This again agrees with out previous result for the case $n = 1$. Following the logic that we used for the $n = 1$ case, we now find

$$\nu = 0.60\ ,\ 0.63\ ,\ 0.65 \tag{23}$$

for $n = 1, 2, 3$. The RG approach thus explains why the values of ν are different for Landau theories that correspond to different values of n, while at the same time, the values of ν are identical for very different models with the same symmetry of the order parameters.

The RG theory can be improved by computing to higher orders in b. In 1973, Bernie Nickel computed the first 6 terms in the RG equation for b in 3 dimensions using Feynman diagrams. Zinn-Justin and Le Guillou developed sophisticated methods to improve the convergence of this perturbation series and resum it. Their analysis gives the results for ν for $n = 1, 2, 3$,

$$\nu = 0.630(2)\ ,\ 0.670(3)\ ,\ 0.705(3)\ . \tag{24}$$

These should be compared to the results quoted earlier from experiment

$$\nu = 0.63\ ,\ 0.67\ ,\ 0.71\ . \tag{25}$$

Estimates of ν obtained by analyzing the high-temperature series for models of magnets on a lattice obtain these same values,

$$\nu = 0.631(3)\ ,\ 0.674(6)\ ,\ 0.711(8)\ . \tag{26}$$

The agreement found for these and the other critical exponents is really quite impressive.

Now, let's take up the last question, involving the lower critical dimensionality. For magnets with $n \geq 2$, I argued these systems have a lower critical dimension at $d = 2$. Can we understand this from the point of view of the RG?

In will now argue that, in the same way that we found the bS^4 term to be a marginal direction in 4 dimensions, field theories of Goldstone bosons have a marginal operator in 2 dimensions. In $(2+\epsilon)$ dimensions, there is an associated fixed point that indicates the appearance of critical behavior at nonzero temperature.

To begin, set up a free energy functional in which the only degrees of freedom are Goldstone bosons. Consider a magnet with $SO(n)$ symmetry. In the low-temperature phase, the expectation value of the order parameter takes the form

$$\langle S^a \rangle = m_0\ (\hat{n})^a \tag{27}$$

where m_0 is a value determined by the thermodynamics and $\hat{n}$ is a unit vector that provides an orietation. The low-energy spectrum of this model is given by fluctuations of the orientation of $\hat{n}$, which are the Goldstone boson modes of oscillation. We should then consider field configurations in which the unit vector $\hat{n}$ varies from point to point. This vector has $(n-1)$ degrees of freedom, corresponding to the $(n-1)$ Goldstone bosons.

Let's try to write a free energy functional for the field $v^a(x)$ that is invariant under the $SO(n)$ symmetry. There are no possible terms with zero derivatives, since any such term would be built from powers of $|\hat{n}|^2(x) = 1$. The only possible 2-derivative term is the kinetic term

$$d^dx\ \frac{1}{2}A(\vec{\nabla}n^a)^2 \tag{28}$$

Other possible structures are constrained by the identity

$$n^a\vec{\nabla}n^a = \frac{1}{2}\vec{\nabla}|\hat{n}|^2 = 0\ . \tag{29}$$

The next possible structures are the 4-derivative terms

$$(\nabla^2 n^a)^2\ , \quad (\vec{\nabla}n^a \cdot \vec{\nabla}n^a)^2 \tag{30}$$

Then the statistical weight $\beta G[S]$ takes the form

$$\int d^dx \Big\{ \frac{1}{2}A(\vec{\nabla}n^a)^2 + \frac{1}{2}B(\nabla^2 n^a)^2 + \cdots \Big\} \tag{31}$$

NLSMweighttwo In this case, since n^a is already normalized, I do not have the freedom to absorb the coefficient A by rescaling n^a. Notice that A is proportional to β or to $1/T$.

Let's consider the behavior of the coefficients under simple dimensional-analysis scaling. Let

$$x = \lambda x' \tag{32}$$

as before. Since n^a is normalized, it cannot be rescaled. Then the weight (33) becomes

$$\int d^d x' \, \lambda^d \left\{ \lambda^{-2} \frac{1}{2} A (\vec{\nabla} n^a)^2 + \lambda^{-4} \frac{1}{2} B (\nabla^2 n^a)^2 + \cdots \right\} \tag{33}$$

The coefficients in this expression transform as

$$\begin{aligned} A &\to \lambda^{d-2} A \\ B &\to \lambda^{d-4} B \end{aligned} \tag{34}$$

The B interaction is irrelevant near 2 dimensions. Terms with more derivatives are more irrelevant. The A interaction is marginal—at least, before considering the effect of interactions. For $d > 2$, the A parameter grows with length scale. Let's then consider the model with the A term only. I will write the statistical weight of this model as

$$\int d^d x \Big\{ \frac{1}{2T} (\vec{\nabla} n^a)^2 \Big\} \tag{35}$$

and it is useful to think about T as the effective temperature of the Goldstone boson system. For reasons that are not especially relevant to our discussion, this expression is called the "nonlinear sigma model".

At zero temperature, the $n^a(x)$ field is frozen to a fixed orientation. I will choose coordinates so that

$$n^a = (0, \cdots, 0, 1)^a \tag{36}$$

Now we can consider fluctuations about this configuration. The fluctuating field $n^a(x)$ can be parametrized as

$$n^a(x) = (\pi^b(x), (1 - \pi^2)^{1/2}) \tag{37}$$

where $\pi^2 = (\pi^b \pi^b)$ and $b = 1, \cdots, n-1$. The $\pi^b(x)$ are the Goldstone boson fields. Differentiating,

$$\vec{\nabla} n^a = \left(\vec{\nabla} \pi^b, -\frac{\pi^b \vec{\nabla} \pi^b}{(1 - \pi^2)^{1/2}} \right) . \tag{38}$$

Then (35) takes the form

$$\int d^d x \, \frac{1}{2T} \Big[(\vec{\nabla} \pi^b)^2 + \frac{(\pi^c \vec{\nabla} \pi^c)^2}{(1 - \pi^2)^{1/2}} \Big] \, .] \tag{39}$$

Notice that there is no mass term for the π^b fields, in accordance with Goldstone's theorem. The second term in brackets is a nonlinear interaction of the π^b fields. Let's treat this by expanding it in powers of π^b. It is also convenient to rescale the π^b variables, $\pi^b \to T^{1/2}\pi^b$, so that we have our canonical normalization of the kinetic term. This brings (39) into the form

$$\int d^dx \left\{ \frac{1}{2}(\vec{\nabla}\pi^b)^2 + \frac{T}{2}(\pi^c\vec{\nabla}\pi^c)^2 \left(1 + T\pi^2 + \cdots\right) \right\} \tag{40}$$

beginP The the parameter T play the role of the coupling constant determining the size of the nonlinear interaction. The original, frozen configuration is the limit $T \to 0$.

For $n = 2$, there is alternative change of variables that gives a simpler result. Let

$$n^a(x) = (\sin\theta(x), \cos\theta(x)) \ . \tag{41}$$

Then

$$\vec{\nabla} n^a = (\cos\theta, -\sin\theta)\, \vec{\nabla}\theta \tag{42}$$

and (35) becomes

$$\int d^dx \ \frac{1}{2T}(\vec{\nabla}\theta)^2 \tag{43}$$

This is a free theory without any nonlinearity. The case $n = 2$ in 2 dimensions has special physics that I will discuss next week. There is no similar transformation for the models with $n \geq 3$. Those models are intrinsically nonlinear and interacting. From here on, I consider those cases only.

I will now work out the RG evolution of the parameter T in 2 dimensions. This argument is due to Alexander Polyakov. As in the previous lecture, I will set up a recursion by dividing the momenta into two groups, high momenta in the interval

$$\frac{\pi}{\lambda a} < |k| < \frac{\pi}{a} \tag{44}$$

and low momenta with $|k| < \pi/\lambda a$. The low-momentum degrees of freedom are contained in a unit vector field

$$\widetilde{n}^a(x) \tag{45}$$

that has only low-momentum Fourier components. This is a unit vector and so obeys

$$\widetilde{n}^a \vec{\nabla} \widetilde{n}^a = 0 \ . \tag{46}$$

Let $e^a(x)$ be a complete basis of unit vectors orthogonal to $\widetilde{n}^a(x)$ at each point. These $e^a(x)$ satisfy

$$e^a_i\, n^a = 0 \qquad e^a_i\, e^a_j = \delta_{ij} \ . \tag{47}$$

By virtue of these relations, the e^a_i satisfy identities such as

$$\begin{aligned} \vec{\nabla}\widetilde{n}^a\, e^a_i + \widetilde{n}^a \vec{\nabla} e^a_i &= 0 \\ \vec{\nabla} e^a_i\, e^a_j + e^a_i \vec{\nabla} e^a_j &= 0 \end{aligned} \tag{48}$$

To describe the Goldstone fluctuations at high momentum, I will introduce fields $\varphi_i(x)$ that have Fourier components only at high monentum and represent the full $n^a(x)$ as

$$n^a(x) = \widetilde{n}^a(x)(1 - (\varphi_i(x))^2)^{1/2} + e_i^a \varphi_i(x) \tag{49}$$

The derivative of $n^a(x)$ is

$$\begin{aligned} \vec{\nabla} n^a = \vec{\nabla}\widetilde{n}^a \ (1 - (\varphi_i(x))^2)^{1/2} - \widetilde{n}^a \, \frac{\varphi_i \vec{\nabla}\varphi_i}{(1 - (\varphi_i(x))^2)^{1/2}} \\ + \vec{\nabla}\varphi_i e_i^a + \varphi_i \vec{\nabla} e_i^a \ . \end{aligned} \tag{50}$$

Squaring (50) and expanding in powers of φ_i, we find

$$\begin{aligned} \frac{1}{2T}(\vec{\nabla} n^a)^2 = \frac{1}{2T}\Big\{ & (\vec{\nabla}\widetilde{n}^a)^2 (1 - \frac{1}{2}(\varphi_i)^2) + (\vec{\nabla}\varphi_i)^2 \\ & + 2\vec{\nabla}\widetilde{n}^a \, \vec{\nabla}(\varphi_i) e_i^a + 2\vec{\nabla}\widetilde{n}^a \, \varphi_i \vec{\nabla}(e_i^a) \\ & + \varphi_i\varphi_j \vec{\nabla} e_i^a \vec{\nabla} e_j^a + 2\phi_i \vec{\nabla}\phi_j \vec{\nabla} e_i^a \, e_j^a + \cdots \Big\} \end{aligned} \tag{51}$$

The last term in the first line here comes from the square of the first term in the second line of (50).

The next step is to integrate over the $\phi_i(x)$. To do this, we take the term

$$\frac{1}{2T}(\vec{\nabla}\phi_i)^2 \tag{52}$$

as giving the zeroth-order statistical weight. Then the contraction of $\phi_i(x)$ fields is

$$\langle \phi_i(x)\phi_j(x) \rangle = \int \frac{d^d k}{(2\pi)^d} \ \frac{T}{k^2} \ , \tag{53}$$

where the integral is taken over the momentum region (44). Evaluating this expression, specifically in 2 dimensions,

$$\langle \phi_i(x)\phi_j(x) \rangle = \delta_{ij} \cdot \frac{T}{2\pi} \log \lambda \tag{54}$$

We can also evaluate

$$\left\langle \vec{\nabla}\phi_i \ \vec{\nabla}\phi_j \right\rangle = \delta_{ij} \ \int \frac{d^2 k}{(2\pi)^2} \ T \frac{k^2}{k^2} = \delta_{ij} \ \frac{T}{4\pi} \frac{1}{a^2} \tag{55}$$

which is a simple number that does not affect the scaling laws, similar to the order b correction to the m^2 term in discussed in the previous lecture.

With this observation, I can integrate over terms in the free energy functional that involve ϕ_i^2 and drop terms that involve $(\vec{\nabla}\phi_i)^2$. This transforms (51) into

$$\frac{1}{2T}\Big\{(\vec{\nabla}\tilde{n}^a)^2(1-\left\langle\phi_i^2\right\rangle)+\langle\phi_i\phi_j\rangle\ \vec{\nabla}e_i^a\vec{\nabla}e_j^a+\cdots\Big\}$$
$$=\frac{1}{2T}\Big\{(\vec{\nabla}\tilde{n}^a)^2(1-(n-1)\mathcal{I}+(\vec{\nabla}e_i^a)^2\ \mathcal{I}\Big\} \tag{56}$$

where

$$\mathcal{I}=\frac{T}{2\pi}\log\lambda\ . \tag{57}$$

To evaluate the second term in (56), introduce a complete set of unit vectors,

$$(\vec{\nabla}e_i^a)^2=(\tilde{n}^a)\vec{\nabla}e_i^a)^2+(e_j^a\vec{\nabla}e_i^a)^2\ . \tag{58}$$

Using the identities above, we can rewrite the first term as

$$(\tilde{n}^a)\vec{\nabla}e_i^a)^2=(-\vec{\nabla}\tilde{n}^a\ e_i^a)^2=(\vec{\nabla}\tilde{n}^a)^2\ . \tag{59}$$

The second term involves the rotation of the e_i^a vectors among themselves from point to point and can be shown to be an irrelevant operator.

In all, the result of the integration out of the $\phi_i(x)$ gives

$$\frac{1}{2T}\Big\{(\vec{\nabla}\tilde{n}^a)^2(1-(n-2)\frac{T}{2\pi}\log\lambda)\Big\}\ . \tag{60}$$

It is convenient to write the correction directly as a correction to T,

$$T\to T(1+\frac{n-2}{2\pi}\ T\log\lambda) \tag{61}$$

Then, considering a continuous process of integration out, we find the RG equation for T in 2 dimensions,

$$\ell\frac{d}{d\ell}T=+\frac{n-2}{2\pi}\ T^2\ . \tag{62}$$

The solution to this equation is

$$T(\ell)=\frac{T(a)}{1-\frac{n-2}{2\pi}\ T(a)\log\ell/a} \tag{63}$$

This equation implies that, however small the original coupling constant $T(a)$ was at short distances in the original model, it grows to become a strong coupling at large distances. This phenomenon is called "asymptotic freedom". It is seen in other condensed matter systems and also in the QCD theory of the strong interactions. Once $T(\ell)$ becomes of order 1, the approximation of keeping only the leading T^2 term in (62) becomes invalid. The apparent divergence of the coupling constant at a finite value of ℓ/a in (63) is an artifact of using this leading order perturbative expression in the regime where the coupling constant has become strong.

The implications of asymptotic freedom for the nonlinear sigma model are quite clear: However low the original value of the temperature, at large distances the model has the same behavior as a high-temperature model in which the Goldstone boson fluctuations completely disorder the spins. There is no phase transition.

Now let's translate this result to the situation in $d = (2 + \epsilon)$. The dimensional-analysis scaling for A above implies the scaling for T

$$\ell \frac{d}{d\ell} T = -(d-2)\, T = -\epsilon\, T \tag{64}$$

Combining this with the effect of the interactions, we find the RG equation for T in $(2 + \epsilon)$ dimensions to be

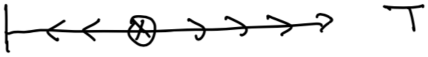
$$\ell \frac{d}{d\ell} T = -\epsilon\, T + \frac{n-2}{2\pi} T^2 \tag{65}$$

This equation has a fixed point at

$$T_* = \frac{2\pi}{n-2}\, \epsilon \tag{66}$$

The fixed point separates the ordered-low temperature phase from the disordered, high-temperature phase.

The fixed point is unstable in the direction of T,



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The rate of the instability is given by taking the derivative of the right-hand side of the RG equation,

$$\frac{d}{dT}(-\epsilon\, T + \frac{n-2}{2\pi} T^2)\Big|_* = (-\epsilon + \frac{n-2}{\pi} T)\Big|_* = \epsilon\ . \tag{68}$$

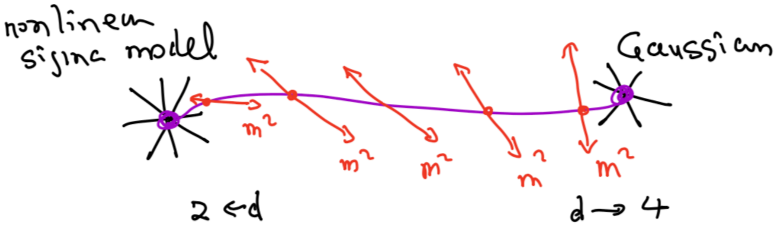
Then we predict

$$\nu = 1/\epsilon + \cdots \approx 1 \qquad \text{in} \quad d = 3 \tag{69}$$

This is very different from the mean field value $\nu = 1/2$. It is larger that the mean-field value and so suggests an approach to the the $(4 - \epsilon)$ value and the experimental value $\nu \approx 0.7$. As before, the value can be improved by computing higher orders in ϵ. At the next order, a dependence on n appears.

The insights from this and the previous lecture give an interesting picture of evolution of the physics of the critical region as the dimensionality of space changes.

The critical point appears at the lower critical dimensionality $d = 2$, where it is well-described by the nonlinear sigma model. As d increases, it becomes a strong-coupling theory that is more difficult to interpret. As d approaches 4, the fixed point becomes well-described by a Landau theory, and finally it disappears into the Gaussian fixed point at $d = 4$. Above $d = 4$, the large-distance theory of the critical point has zero nonlinear interations. Thus we find mean-field exponents for all dimensions greater than 4. Thus, the models with $n \geq 2$ have $d = 2$ as a lower critical dimension and $d = 4$ as an upper critial dimension.



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In $d = 2$, for $n = 2$, the physical picture at large distances must be very different from what I have described about for $n > 2$. I will discuss this special case in the next lecture.