Nov. 28

Physics 212 – Statistical Mechanics

The Renormalization Group and Critical Exponents - 1

In the previous lecture, we constructed scaling transformations in magnetic models based on the principle of "integrating out" degrees of freedom. In principle, these transformations could take into account the full dynamics of interacting spins while insuring that the phsics of the model on large length scales would be preserved. It was not so easy, though, to carry out these transformations exactly. In this lecture, I will discuss another approach to integrating out, based on fields on a continuum. This turns out to be the method that leads to more definite conclusions about the computation of critical exponents and the origin of universality classes.

To introduce this method, I will first think about the change of scale for continuum fields without taking account of interactions. The results will agree with simple dimensional analysis. Then we will to add the interactions back systematically using perturbation theory. We will see that this has the potential to change the scaling exponents and even to determine them uniquely from theory.

For most of this lecture, I will work with the Landau theory associated with the Ising model. This contains one order parameter, the continuum field $S(x)$. I will impose the Z_2 symmetry of the model.

If we view the free energy of the model as a statistical weight for the field, we need to compute functional integrals of the form

$$\int \mathcal{D}S \; exp\Big[-\int d^dx \,\Big\{\frac{1}{2}(\vec{\nabla}S)^2 + \frac{1}{2}m^2S^2 + \frac{b}{4}S^4 + \frac{c}{6}S^6 \cdots\Big\}\Big] \tag{1}$$

As in our discussion of scaling, I have absorbed the factor of β in front of $G[S]$ into the normalization of the field $S(x)$. This is reasonable to do, because in the critical region $\beta \approx 1/T_c$, which is a finite number, neither zero nor infinity. I will use the canonical normalization

$$-\int d^dx \,\Big\{\frac{1}{2}(\vec{\nabla}S)^2 + \cdots\Big\} \tag{2}$$

as a touchstone to organize how we make a change of scale. This convention implies that, to the first approximation, the Green's function of the S field has the simple form

$$\langle S(x)S(0)\rangle = \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2} \tag{3}$$

Remember that the mass terms in the Green's function will be small near $\beta = \beta_c$. where $G_0(x, y)$ is the Green's function associated with $(-\nabla^2)$. I will work in a general spatial dimension d.

Begining with a statistical weight of this type, I would like to to remove degrees of freedom by integrating out. Remember that a continuum field always has divergences that must be defined using an underlying atomic length scale a as a cutoff. Momentum integrals are not taken over an infinite range but rather over

$$\int_{-\pi/a}^{\pi/a} \frac{dk}{(2\pi)} \tag{4}$$

With this starting point, I will remove degrees of freedom at high momentum or short distances, moving the spatial cutoff to λa and the upper limit of integration in momentum to $\pi/\lambda a$. We can then make the scale transformation

$$x = \lambda x' \qquad k = k'/\lambda \ , \tag{5}$$

so that the cutoff returns to its original position and the integrals over k again go up to π/a. This defines the recursion. For simplicity, I will use a spherically symmetric cutoff on the momentum range

$$\int_{|k|<\pi/a} \frac{d^d k}{(2\pi)^d} \tag{6}$$

The recursion then proceeds in two steps. First, we integrate over Fourier components of the fields with momenta in the range

$$\frac{\pi}{\lambda a} < |k| < \frac{\pi}{a} \tag{7}$$

These degrees of freedom are coupled to degrees of freedom at lower momenta through the nonlinear terms in the expression for the free energy. Then we make the scale transformation (5) to return the momentum cutoff to its original position. At the same time, we rescale the field $S(x)$ so that our convention (2) on the coefficient of the kinetic term for $S(x)$ is restored. These two steps define a recursion formula of the sort that we discussed in the previous lecture. Here, however, we can argue more clearly that the recursion is taking place in the full space of all possible local interactions of $S(x)$.

To analyze this recursion formula, I will first consider the consequences of only the second step of the recursion, the rescaling of distances and of the normalization of $S(x)$. I will ignore the nonlinear couplings, so the high-momentum degrees of freedom simply drop out at each step. Once we understand the consequences of this step, I will add back the nonlinear interactions and see what changes result from allowing the high-momentum degrees of freedom to interaction with degrees of freedom at ordinary momenta.

Let's now analyze the consequences of the scale transformation (5). This transformation rescales

$$\frac{\partial}{\partial x} = \frac{1}{\lambda}\frac{\partial}{\partial x'} \qquad S = \lambda^{-\alpha} S' \ , \tag{8}$$

where α is a constant to be determined.

This transformation alters the free energy expression given above. This expression changes to

$$\int d^dx' \lambda^d \left\{ \lambda^{-2-2\alpha}\frac{1}{2}(\vec{\nabla}' S')^2 + \lambda^{-2\alpha}\frac{1}{2}m^2 S'^2 + \lambda^{-4\alpha}\frac{b}{4}S'^4 + \lambda^{-6\alpha}\frac{c}{6}S'^6 + \cdots \right\} \tag{9}$$

We fix α by the criterion tht the kinetic term is normalized with the coefficient 1 as in (2). This gives

$$\alpha = (d-2)/2 \tag{10}$$

This is the result that we found for the scaling dimension of the spin operator in our scaling discussion by using dimensional analysis. The other results of the rescaling also correspond to those of dimensional analysis. The other coefficients in the statistical weight tranform as follows:

$$\begin{aligned} m^2 &\to \lambda^{d-2\alpha}\ m^2 = \lambda^2\ m^2 \\ b &\to \lambda^{d-4\alpha}\ m^2 = \lambda^{4-d}\ b \\ c &\to \lambda^{d-6\alpha}\ m^2 = \lambda^{6-2d}\ c \end{aligned} \tag{11}$$

In $d = 4$, only the m^2S^2 term grows under this transformation. The coefficient of the S^4 operator remains constant, and the other coefficients decrease. You should note that the coefficients of operators with higher numbers of derivatives also decrease, for example,

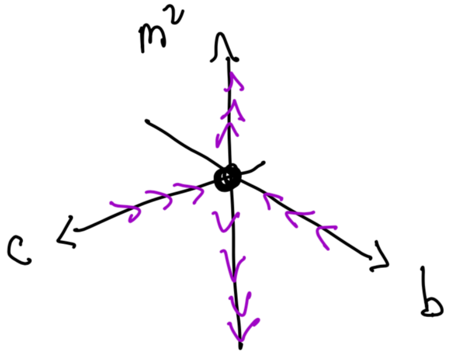
$$\begin{aligned} \eta(\vec{\nabla}S)^4 &\to \lambda^d\lambda^{-4-4\alpha}\eta(\vec{\nabla}S)^4 = \lambda^{-d}\eta(\vec{\nabla}S)^4 \\ \zeta(\vec{\nabla}S)(\nabla^2\vec{\nabla}S) &\to \lambda^d\lambda^{-4-2\alpha}\zeta(\vec{\nabla}S)(\nabla^2\vec{\nabla}S) = \lambda^{-2}\zeta(\vec{\nabla}S)(\nabla^2\vec{\nabla}S) \end{aligned} \tag{12}$$

We can write these dependences as evolution equations in the space of coupling constants,

$$\begin{aligned} &\ell\frac{d}{d\ell}m^2(\ell) = +2\ m^2(\ell) \\ &\ell\frac{d}{d\ell}b(\ell) = +(4-d)\ b(\ell) \qquad \ell\frac{d}{d\ell}c(\ell) = +(6-2d)\ c(\ell) \\ &\ell\frac{d}{d\ell}\eta(\ell) = -d\ \eta(\ell) \qquad \ell\frac{d}{d\ell}\zeta(\ell) = -2\ \zeta(\ell) \end{aligned} \tag{13}$$

The evolution under these equations is called the *Renormalization Group flow* or *RG flow.* Here, the RG flow is given only by the effects of rescaling, but soon we will write RG flows that also include the effects of interactions.

It is interesting to imagine the solutions of the equations (13) as literally a flow in the space of couplings. The simplest case to visualize is $d > 4$. In that case, all couplings decrease except for m^2. We have



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In such high dimensions — as long as the original free energy flows into the region of small couplings under the RG flows — the nonlinear terms all go to zero at large distances. Only the m^2S^2 term is growing in importance. For $d < 4$, the bS^4 also grows. I will discuss its behavior later in the lecture.

I have now introduced a picture that is amazingly powerful in understanding complex statistical mechanical systems, and also in understanding quantum field theories. This picture was developed by Kenneth Wilson and is often called the *Wilsonian viewpoint*. We view individual theories as points in a large space, the *space of all possible interactions*. Under the RG flow, models are transformed into on another. By visualizing these flows and following them up to large distance scales, we can work out the qualitative behavior of any model.

Consider, for example, the most general local free energy functional that we can write for the field $S(x)$. This expression has an infinite number of parameters, so the space of possible interactions is infinite-dimensional. However, in some finite region of the parameter space, it is permissible to enumerate these interactions by writing a Taylor series in $S(x)$ and $\vec{\nabla}$. Let's consider the effect on such models of the flows worked out above — still ignoring all nonlinear interactions. In $d > 4$, almost every term in this Taylor series flows to zero as we run the RG flow to large distance scales by integrating out more and more degrees of freedom. Asymptotically, we approach to free energy functional

$$g[S] = \frac{1}{2}(\vec{\nabla}S)^2 + \frac{1}{2}(\ell/a)^2 m^2 S^2 \ . \tag{15}$$

For $m^2 = 0$, we have the scale-invariant theory

$$g[S] = \frac{1}{2}(\vec{\nabla}S)^2 \ . \tag{16}$$

This is a fixed point of the RG, called the *Gaussian fixed point.* For $m^2 > 0$ or $m^2 < 0$, we move away from the fixed point to a different theory dominated by the effects of the $m^2 S^2$ term. Then the models with $m^2 = 0$ correspond to the critical temperature T_c. For $m^2 > 0$, we move to models with large m^2 and thus short correlation length. These are models in the high-temperature, disordered, phase of the magnet. For $m^2 < 0$, we move to models with a strong instability toward spontaneous symmetry breaking. When m^2 becomes large and negative, considerations of stability tell us that we can no longer ignore the bS^4 term and perhaps other terms also. The description of both of these regimes at large m^2 is outside the scope of this analysis. But, this analysis does predict the rate of growth of m^2

$$m^2(\ell) = (\ell/a)^2 m^2 \tag{17}$$

We can use $|m^2(\ell)| \sim 1$ to estimate the correlation length and the value of the exponent ν, as we did in the previous lecture. Models with small m^2 are close to T_c, so write

$$m^2 = ct \tag{18}$$

Then

$$m^2(\ell) = c(\ell/a)^2 \ t \ , \tag{19}$$

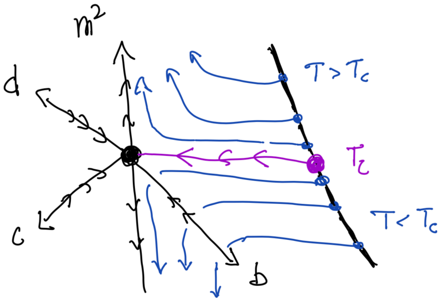
The length scale ξ is found where $m^2(\ell)$ is of order 1,

$$\xi \sim \ell \qquad \text{with} \qquad c(\ell/a)^2 \ t \sim 1 \ , \tag{20}$$

Eliminating ℓ,

$$\xi \sim t^{-1/2} \qquad \text{or} \qquad \nu = \frac{1}{2} \tag{21}$$

Notice that we did not need to make any specific assumptions about the form of the original model at the original atomic spacing. The only assumption is that this model transforms into a model with relatively small couplings under the RG flow. The starting point could have been a lattice model of magnetism, an alloy, a fluid. Here is a picture of flows for such a general initial model:



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The dark line in the figure gives the line of the original model (say, the Ising model on a lattice or the a real liquid-gas system near its critical point) as a function of temperature T. My assumption is that one point on this line flows into the Gaussian fixed point. This is very plausible, because all directions around the Gaussian fixed point are attractive except for the m^2S^2 direction. This picture predicts mean-field exponents $\nu = 1/2$, $\eta = 0$ for a wide variety of models (and, maybe, *all models*) with a single order parameter and Z_2 symmetry. This is "universality", in the sense of the earlier lecture on critical exponents.

Wilson introduced the following nomenclature for the behavior of operator coefficients in a general free energy functional in the vicinity of a fixed point of the RG. Let $g_*[S]$ be the fixed point free energy. Then $g_*[S] \to g_*[S]$ under an RG transformation. Nearby points in the space of all interactions are linear deviations from $g_*[S]$,

$$g[S] = g_*[S] + c_i \delta g_i[S] \; . \tag{23}$$

Since the point $c_i = 0$ is transformed into itself, the RG must transform the functionals in this class by

$$\ell \frac{d}{d\ell} c_i = A_{ij} c_j + \mathcal{O}(c^2) \; . \tag{24}$$

Now choose a basis in which the matrix A_{ij} is diagonal. In this basis, the individual c_i transform as

$$\ell \frac{d}{d\ell} c_i = \alpha_i c_i + \mathcal{O}(c^2) \; . \tag{25}$$

Then

$$c_i(\ell) = (\ell/a\Big)^{\alpha_i} c_i \tag{26}$$

If we write as our initial condition

$$G[S] = \int d^dx \; \Big\{ g_*[S] + \sum_i c_i \mathcal{O}_i \Big\} \tag{27}$$

then, under a rescaling $a \to \lambda a$, the Gibbs' free energy transforms as

$$G[S] \to \int d^dx \lambda^d \Big\{ \lambda^{-d} g_*[S] + \sum_i \lambda^{\alpha_i} c_i \mathcal{O}_i \Big\} \; . \tag{28}$$

Then the operator $\mathcal{O}_i$ transforms as

$$\mathcal{O}_i \to \lambda^{-D_i} \mathcal{O}_i \tag{29}$$

where

$$D_i = d - \alpha_i \tag{30}$$

Operators of dimension $D < d$ grow in importance at large distances. Operators with $D > d$ shrink in importance at large distances. In terms of the coefficients c_i, the scaling is

$$c_i(\ell) = (\ell/a)^{\alpha_i} c = (\ell/a)^{d - D_i} \tag{31}$$

where a is the original atomic spacing. This argument can be applied to the RG transformation coming from dimensional analysis that we are doing here, but it is important to realize that it applies also to any more genera RG transformation, including those that includes the effects of interactions.

Wilson classified the operators near a fixed point as follows:

- **Relevant** operators: $D_i < d$ or $\alpha_i > 0$. These are unstable directions about the fixed point, leading to increasing importance of the operator $\mathcal{O}_i$ at large scales.
- **Irrelevant** operators: $D_i > d$ or $\alpha_i < 0$. These are stable directions about the fixed point, leading to decreasing importance of the operator $\mathcal{O}_i$ at large scales.
- **Marginal** operators: $D_i = d$ or $\alpha_i = 0$. These are directions of neutral stability, which might become stable or unstable when higher-order effects are taken into account. It is also possible to have an *exactly marginal* direction with $alpha_i = 0$ to all orders.

Only *relevant* and *marginal* directions contribute to the large-distance behavior of models near $T = T_c$. In most examples, there are only a finite number of these directions in the infinite-dimensional space. Models that differ only by *irrelevant* operators have the same large-distance behavior and belong to the same *universality class*.

Our analysis of the Gaussian fixed point is correct for the case of $b = c = \cdots = 0$ in which all nonlinear interactions vanish. However, when b is turned on, the situation changes. I will now investigate this, introducing a small b and working out its consequences.

We now need to discuss how to compute correlation functions such as $\langle S(x)S(y)\rangle$ for $b \neq 0$. I will do this using perturbation theory, assuming that b is small. We would like to write $\langle S(x)S(y)\rangle$ as a Taylor expansion in powers of b. Take as the starting point the statistical weight

$$\int \mathcal{D}S \ \exp\Big[-\int d^dx \ \Big\{\frac{1}{2}(\vec{\nabla}S)^2 + \frac{1}{2}m^2S^2 + \frac{b}{4}S^4\Big\}\Big] \ . \tag{32}$$

Expand the exponential

$$\begin{aligned}\int \mathcal{D}S \ \exp\Big[-\int d^dx \ \frac{1}{2}(\vec{\nabla}S)^2\Big] \\ \cdot\Big(1 - \frac{m^2}{2}\int d^dx \ S^2 + \frac{1}{2}\Big(\frac{m^2}{2}\int d^dx \ S^2\Big)^2 + \cdots\Big) \\ \cdot\Big(1 - \frac{b}{4}\int d^dx \ S^4 + \frac{1}{2}\Big(\frac{b}{4}\int d^dx \ S^4\Big)^2 + \cdots\Big)\end{aligned} \tag{33}$$

Notice that, since the parameter m^2 is small in the critical region, I am also considering the $m^2 S^2$ term as a perturbation. If we use this expression to compute correlation functions, we obtain expressions with a Gaussian weight in exponent and many powers of $S(x)$. Fortunately, we can evaluate these automatically using Wick's theorem. To begin, I remind you that the contraction of $S(x)$ field is given by

$$\langle S(x)S(y)\rangle = \wick{\c S(x) \c S(y)} = G_0(x,y) \tag{34}$$

where $G_0(x,y)$ satisfies

$$(-\nabla^2)\ G_0(x,y) = \delta(x-y) \tag{35}$$

(Remember that I have scaled the factor of T into the normalization of $S(x)$.) The Fourier space representation of this function is

$$\wick{\c S(x) \c S(y)} = G_0(x,y) = \int \frac{d^dk}{(2\pi)^d}\ e^{ik\cdot(x-y)}\ \frac{1}{k^2}\ . \tag{36}$$

Let's now compute the spin-spin correlation function up to terms of order b^1, ignoring m^2. This is given by expanding

$$\langle S(x)S(y)\rangle = \frac{\int \mathcal{D}S e^{-\int d^dx\, \frac{1}{2}(\vec{\nabla}S)^2} S(x)S(y)(1-\frac{b}{4}\int d^z\ S^4)}{\int \mathcal{D}S e^{-\int d^dx\, \frac{1}{2}(\vec{\nabla}S)^2}(1-\frac{b}{4}\int d^z\ S^4)}\ . \tag{37}$$

For the moment, I will concentrate on the numerator

$$\int \mathcal{D}S e^{-\int d^dx\, \frac{1}{2}(\vec{\nabla}S)^2} S(x)S(y)(1-\frac{b}{4}\int d^z\ S^4)\ . \tag{38}$$

We can evaluate this as a sum of contractions. One contraction is

$$-\frac{b}{4}\int d^dz\ \wick{\c1 S(x) \c2 S(y) \quad \c1 S(z) \c2 S(z) \c3 S(z) \c3 S(z)} \tag{39}$$

The value of this contraction is

$$-\frac{b}{4}\int d^dz\ \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d} e^{i(k\cdot(x-z)+p\cdot(y-z)+q\cdot(z-z)}\frac{1}{k^2p^2q^2} \tag{40}$$

The integral over d^dz gives

$$\int d^dz e^{i(k\cdot(x-z)+p\cdot(y-z)+q\cdot(z-z)} = (2\pi)^d\delta^{(d)}(k+p)e^{ik\cdot(x-y)} \tag{41}$$

There are $4\cdot 3$ different contractions that give this result. Keeping only these terms, we find

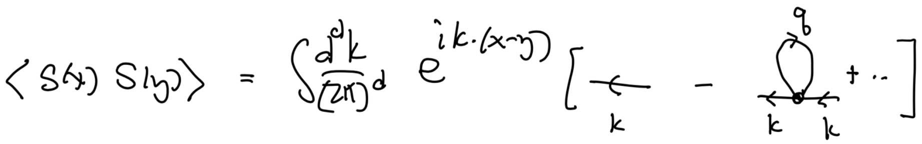
$$\langle S(x)S(y)\rangle = \int \frac{d^dk}{(2\pi)^d}e^{ik\cdot(x-y)}\Big(\frac{1}{k^2}-\frac{1}{k^2}\Sigma\frac{1}{k^2}\Big)\ , \tag{42}$$

where

$$\Sigma = 12 \cdot \frac{b}{4} \cdot \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2} \tag{43}$$

If we integrate out one layer, the integral will be taken over a spherical shell in momentum space $\pi/\lambda a < |q| < \pi/a$. Notice that Σ is a number; it does not depend on the momentum k.

It builds intuition to represent each term in (42) as digram in which each Green's function is represented by a line carrying the associated momentum,

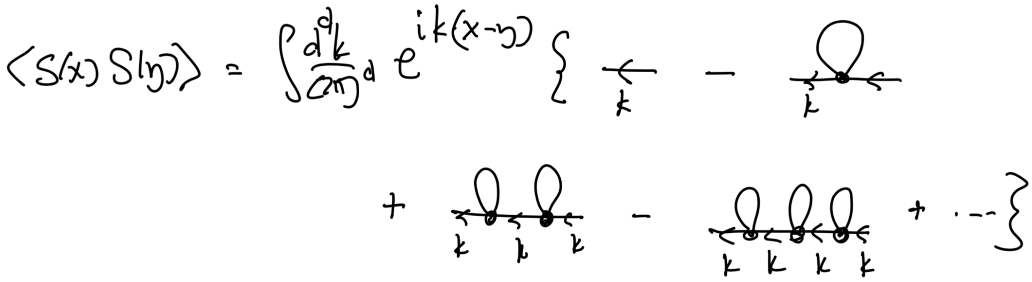


(44)

The b term is represented by a vertex connecting 4 of these lines. The d^dz integrals impose momentum conservation at each vertex.

These diagrams are called "Feynman diagrams" after their inventer, Richard Feynman. You are probably aware that Feynman diagrams are the major tool in quantum field theory. Long, difficult books are written about their properties. Here I will dip into the theory of Feynman diagrams only far enough to reach the goals of this lecture. Feynman diagrams can be constructed for fields or particles of any spin. Here we will deal only with the simplest case of scalar (spin-0) fields.

Using the intuition from Feynman diagrams, it is easy to visualize contributions to the spin-spin correlation function at higher orders in b. If we consider the term of order b^n and consider contractions that link successive bS^4 terms, we find the following set of diagrams



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The evaluation of these diagrams gives

$$\langle S(x)S(y)\rangle = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-y)} \left\{ \frac{1}{k^2} + \frac{1}{k^2}\Sigma\frac{1}{k^2} + \frac{1}{k^2}\Sigma\frac{1}{k^2}\Sigma\frac{1}{k^2} + \cdots \right\} \tag{46}$$

where Σ is the integral written in (43). This sum is actually a geometric series, so we can sum it up,

$$\langle S(x)S(y)\rangle = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-y)} \frac{1}{k^2} \frac{1}{1+\Sigma/k^2}$$

$$= \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-y)} \frac{1}{k^2+\Sigma} \ . \tag{47}$$

The effect of these diagrams is, then, to modify the Green's function equation. If we had not treated m^2 as a perturbation, the last term under the integral would read

$$\frac{1}{k^2+m^2+\Sigma} \tag{48}$$

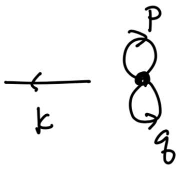
Then we see that Σ is a small shift of the m^2 term that is induced by the nonlinear interaction bS^4. This just means that the value of T_c that we predicted from Landau theory is not actually the correct T_c in the presence of interactions. This is no surprise. To account for Σ, we need to choose another temperature in the original theory such that the original m^2 and the corrections due to the interaction cancel and the correlation function $\langle S(x)S(y)\rangle$ returns to the form

$$\langle S(x)S(y)\rangle = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot(x-y)} \frac{1}{k^2} \ . \tag{49}$$

Another contraction contributing to (38) is

$$-\frac{b}{4}\int d^dz\ \overline{S(x)S(y)} \quad \overline{S(z)S(z)}\,\overline{S(z)S(z)} \tag{50}$$

This corresponds to the Feynman diagram



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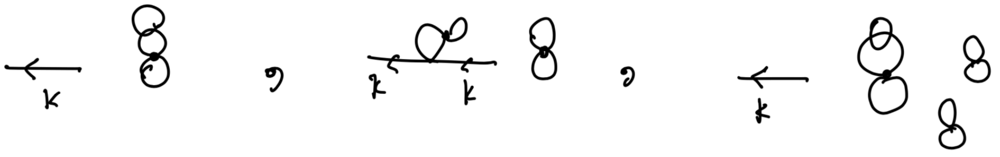
There are more "disconnected" contributions of this type,



(52)

However, it can be shown that the disconnected contributions always cancel against the contributions from the denominator of (37),

(53)

You might remember that we saw such a cancellation between numerator and denominator when we computed the spin-spin correlation function in the high-temperature expansion of the Ising model. It can be shown that, in these continuum problems, the cancellation is exact and there are no left-over terms representing excluded volumes. Thus, I will ignore disconnected diagrams in the rest of this discussion.

Now I would like to turn to the calculation of $\mathcal{O}(b)$ corrections to the four-point correlation function

$$\langle S(x_1)S(x_2)S(x_3)S(x_4)\rangle \tag{54}$$

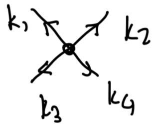
This gives the effect that is most interesting for us for the theory of critical exponents. In $\mathcal{O}(b^0)$, Wick's theorem gives

$$\begin{aligned}&\langle S(x_1)S(x_2)S(x_3)S(x_4)\rangle\\ &\quad = G_0(x_1,x_2)G_0(x_3,x_4) + G_0(x_1,x_3)G_0(x_2 3,x_4) + G_0(x_1,x_4)G_0(x_2,x_3)\ ,\end{aligned} \tag{55}$$

and there is no contribution that connects all four of the x_a. The first fully connected contribution comes in $\mathcal{O}(b)$,

$$\left\langle S(x_1)\,S(x_2)\,S(x_3)\,S(x_4)\ \left(-\frac{b}{4}\int d^dz\ S^4(z)\right)\right\rangle$$

$$= \int d^dz\ \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_4}{(2\pi)^d}\ e^{+ik_1(x_1-z)}e^{+ik_2(x_2-z)}e^{+ik_3(x_3-z)}e^{ik_4\cdot(x_4-z)}$$

$\cdot$ 

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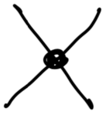
The value of this contribution is

$$\int\frac{d^dk_1}{(2\pi)^d)}\frac{d^dk_2}{(2\pi)^d)}\frac{d^dk_3}{(2\pi)^d)}\frac{d^dk_4}{(2\pi)^d)}e^{ik_1\cdot x_1}e^{ik_2\cdot x_2}e^{ik_3\cdot x_3}e^{ik_4\cdot x_4}\frac{-6b}{k_1^2k_2^2k_3^2k_4^2}(2\pi)^d\delta^{(d)}(k_1+k_2+k_3+k_4) \tag{57}$$

Notice that the integral over z produces a momentum-conserving delta function. To keep the expressions to come relatively simple, I would like you to realize that most of the factors in this expression are obvious, so that I can abbreviate it as

$$\mathrm{FT}\left\{\frac{-6b}{k_1^2k_2^2k_3^2k_4^2}\right\}\ , \tag{58}$$

where FT indicates the Fourier transform with a momentum-conserving delta function. In fact, the expressions for the external lines are also fixed, so finally I will represent this contribution as simply



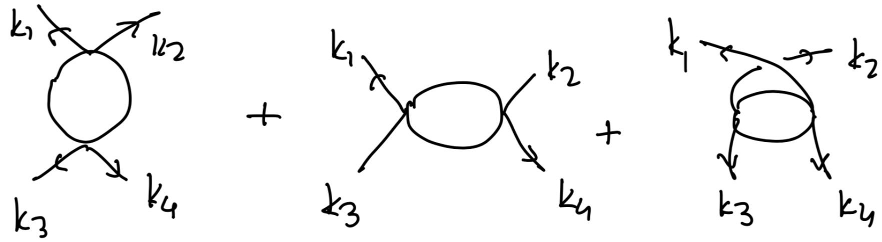
$$= -6b \tag{59}$$

I would like to spend a little time discussing the factor of 6. In the expression (38), the factor that appears is $(b/4)$. When we make Wick contractions with the external points, there are 4! ways to contract with the 4 S's in $\int d^dzS^4$. Thus we find the factor

$$4! \cdot \frac{1}{4} = 6 \tag{60}$$

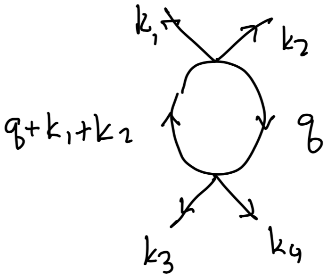
You will recall, however, that in the expression (43) what we found was a factor $12/4 = 3$. The reason for this was that the number of contractions with S^4 with itself was smaller than 4! by a factor of 2, because two of the S's were contracted with one another. This is associated with a symmetry of the Feynman diagram with a loop in (44): We can unplug the loop, reverse its direction, and plug it back in without changing the diagram. In the following, I will account each S^4 vertex by the factor $(-6b)$ and tell you explicitly where such symmetries of Feynman diagrams reduce the final result by a factor of 2.

After this long introduction, we are ready to discuss the interaction corrections of the S^4 operator. We can compute these from the modification of the 4-spin correlation function of $\mathcal{O}(b^2)$. I will ignore disconnnected diagrams and diagrams that only modify the external lines. Then there only 3 contributions, corresponding to the Feynman diagrams



(61)

Let's evaluate the first of these diagrams carefully. Showing the momentum flow in detail, the diagram is



(62)

In the abbreviated notation of (59), the value of this diagram is

$$\frac{1}{2}(-6b)^2 \int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+k_1+k_2)^2} \tag{63}$$

The factor of 1/2 is a symmetry factor for this diagram, since we can unplug, the two lines in the loop, cross them, and plug then in again without changing the diagram. The integral will be taken over a spherical shell in momentum space $\pi/\lambda a < |q| < \pi/a$. In this case, the integral does depend on the values of the external momenta.

I would like to concentrate on the value of this integral in $d = 4$. In this case, the integral is logarithmic and we can estimate its value as

$$\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+k_1+k_2)^2} \approx \int_{\pi/\lambda a}^{\pi/a} dq\, q^3\ \mathcal{A}(4)\ \frac{1}{q^4} \tag{64}$$

Since the momentum states k_1 and k_2 are not integrate out, these momenta are small compared to q. The area of the unit sphere in 4 dimension is $\mathcal{A}(4) = 2\pi^2$. Then the above equals

$$\frac{2\pi^2}{16\pi^4}\int_{\pi/\lambda a}^{\pi/a}\frac{dq}{q} = \frac{1}{8\pi^2}\log\lambda \tag{65}$$

The other two diagrams in (61) have the same value as (65). The effect of these three diagrams, then, is to give a shift of b,

$$-6b \to -6b + 3\cdot(18b^2)\frac{1}{8\pi^2}\log\lambda \tag{66}$$

This contributes to the RG flow in 4 dimensions

$$\ell\frac{d}{d\ell}b(\ell) = -\frac{9}{8\pi^2}b^2 \tag{67}$$

In (11), we saw that bS^4 was a marginal operator in 4 dimensions. The calculation resolves the ambiguity and tells us that, when interactions are taken into account, bS^4 is actually mildly irrelevant in 4 dimensions.

Actually, we are interested in the scaling of the operator bS^4 in dimensions that we can actually realize in experiment, $d = 3$ and maybe also $d = 2$. As a step toward that, let's think about the situation in dimensions just slightly less than 4. Write

$$d = 4 - \epsilon \tag{68}$$

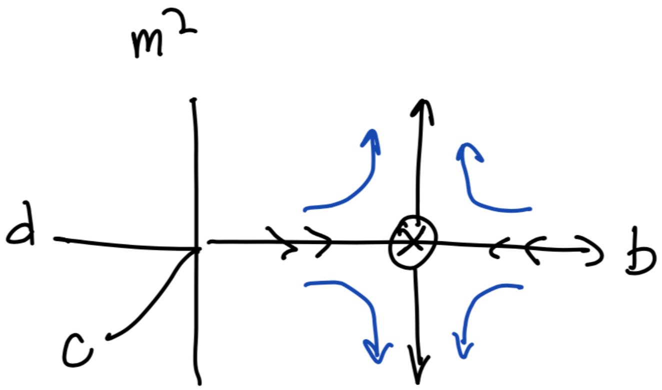
Then the operator bS^4 gets a positive scaling contribution from dimensional analysis, as in (11), and a negative contribution from the effect of interactions computed in (67). Putting these together, we find

$$\ell\frac{d}{d\ell}b(\ell) = +\epsilon\ b - \frac{9}{8\pi^2}b^2 \tag{69}$$

binfour The coefficient $b(\ell)$ given by the RG flow increases for small b but decreases when b is large enough. So there is a fixed point of the RG flow at

$$b_* = \frac{8\pi^2}{9}\,\epsilon \tag{70}$$

Putting this together with the fact that $m^2(\ell)$ grows along the RG flow, we find the flow diagram in dimensions d just less than 4 as

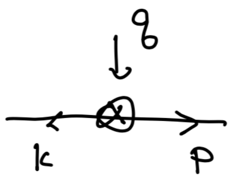


(71)

This fixed point is called the Wilson-Fisher fixed point. It was discovered by Kenneth Wilson and Michael Fisher in 1972. For d slightly less than 4, the Wilson-Fisher fixed point dominates the Gaussian fixed point. It controls the large-distance behavior near T_c for all models that are attracted to the region of weak coupling. Wilson and Fisher conjectured that this fixed point continues to dominate when we are well below 4 dimensions, for example, at $d = 3$. Wilson actually carried out a non-perturbative search for additional fixed points of the RG flow in this set of theories and did not find any. Thus, it is possible that generic models in this space flow to the Wilson-Fisher fixed point for $d < 4$.

We can also use Feynman diagrams to compute the effect of b on the scaling of the operator m^2S^2. The lowest-order correlation function that includes S^2 is

$$\langle S(x)\, S(y)\;\; S^2(z)\rangle \;=$$

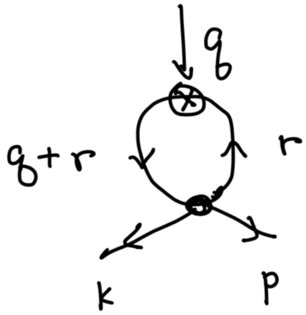


(72)

Here q is the Fourier transform variable associated with z, i.e., the momentum squirted into the S^2 operator. The value of this diagram in our abbreviated notation is

$$\mathrm{FT}\left\{\frac{1}{k^2}\cdot 2\cdot\frac{1}{p^2}\right\} \qquad \text{or just} \qquad 2\;. \tag{73}$$

This correlation function receives a correction in order b^1 from the Feynman diagram



(74)

The value of this diagram, in the abbreviated notation, is

$$2\cdot\frac{1}{2}(-6b)\int\frac{d^dr}{(2\pi)^d}\frac{1}{r^2}\frac{1}{(q+r)^2} \tag{75}$$

Note again a symmetry factor of 1/2. The integral here is the same one that we computed in (64). We find, then, that the correlation function is shifted by

$$2\to 2\cdot\left(1-3b\frac{1}{8\pi}\log\lambda\right) \tag{76}$$

This implies that the S^2 operator coefficient scales slightly differently from numerical analysis,

$$\ell\frac{d}{d\ell}m^2=(2-\frac{3b}{8\pi^2})\ m^2 \tag{77}$$

At the fixed point,

$$\ell\frac{d}{d\ell}m^2=(2-\frac{1}{3}\epsilon)\ m^2 \tag{78}$$

Then the unstable m^2 direction at the fixed point grows as

$$m^2(\ell)=(\ell/a)^{2-\epsilon/3}\ m^2 \tag{79}$$

This result implies

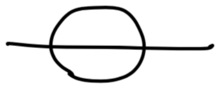
$$\xi\sim\ell\qquad\text{for}\qquad(\ell/a)^{2-\epsilon/3}\ t\sim 1 \tag{80}$$

or

$$\xi\sim t^{-1/(2-\epsilon/3)}\qquad\text{or}\qquad\nu=\frac{1}{2-\epsilon/3} \tag{81}$$

to the leading order in $eps=(4-d)$.

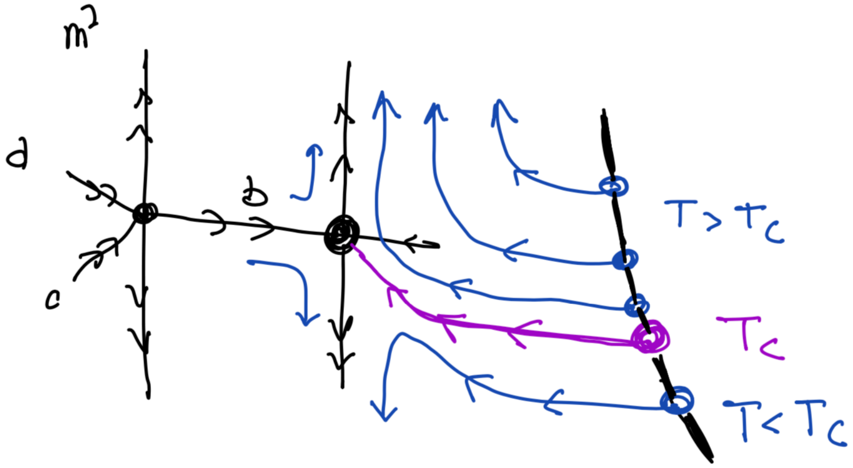
I have not analyzed the corrections to the exponent η in powers of ϵ. It turns out that the first correction to η comes from the diagram



(82)

This diagram is of order $b^2 \sim \epsilon^2$ and so the correction to $\eta = 0$ is always predicted to be small in an analysis in powers of ϵ. This actually explains the fact that η is always measured to be a small number in 3 dimensions.

We then come to the following picture: Within the space of all possible free energy functionals, there is a region attracted to the Wilson-Fisher fixed point. The RG flows have the topology



$$(83)$$

The fixed point is unstable along the m^2S^2 direction. The rate of the instability gives the critical exponent ν found in (81). Naively setting $\epsilon = 1$ for 3 dimensions and ignoring possible corrections of order ϵ^2, we find

$$\nu = 0.60 \tag{84}$$

This should be compared to $\nu = 0.5$ for Landau theory and $\nu = 0.63$ for the best experimental values. The prediction of this theory is applies to a wide variety of underlying models whose common feature is that they have one order parameter field. Clearly, we are making progress toward a general theory of the critical exponents,.