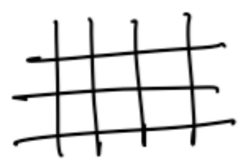
Nov. 14

Physics 212 – Statistical Mechanics

The Migdal Recursion Formula

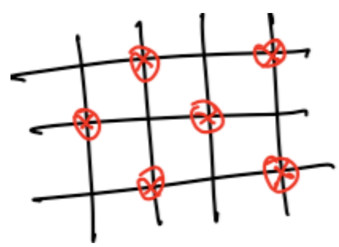
At the end of the previous lecture, I explained that we need a method for carrying out scale transformations in a statistical mechanical model that can take into accout the effects of the nonlinear interactions in the model. In this lecture, I will introduce methods of this type to analyze lattice models of magnetism.

I will take as my example the 2-dimensional Ising model on a square lattice



(1)

To analyze this model, we can try to reduce the lattice spacing by summing over the spins on *odd* lattice sites



(2)

This gives an Ising model with new interactions on a square lattice with spacing $\sqrt{2}$. In principle, we could solve the model completely if we could rigorously carry out this transformation an arbitrarily large number of times.

Write the Ising model partition function as we did earlier

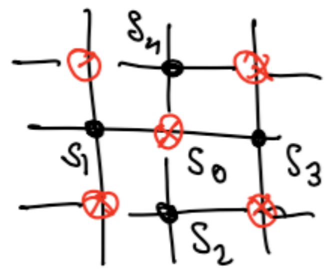
$$Z = \sum_S \prod_{bonds} (2\cosh\beta J)(1 + zS_iS_{i+\nu}) \tag{3}$$

where

$$z = \tanh\beta J \tag{4}$$

All formulae will depend on the combination βJ, so, for simplicity, set $J = 1$.

At the first stage, we must do a spin sum over the eliminated spin S_0 with the neighboring spins S_1, S_2, S_3, S_4 fixed,



(5)

This sum is

$$\frac{1}{2}\sum_{S_0}(1+zS_0S_1)(1+zS_0S_2)(1+zS_0S_3)(1+zS_0S_4)$$
$$=1+z^2(S_1S_2+S_2S_3+S_3S_4+S_4S_1)$$
$$+z^2(S_1S_3+S_2S_4)+z^4S_1S_2S_3S_4 \tag{6}$$

This transformation generates an interaction of the original form plus two new types of interactions, a next-nearest-neighbor interaction and a 4-spin interaction. Since this calculation is only illustrative, I will simplify it by making some (unjustified) assumptions. Let's, then, drop the additional interactions and approximate the sum given above by

$$(1+z^2S_1S_2)(1+z^2S_2S_3)(1+z^2S_3S_4)(1+z^2S_4S_1) \tag{7}$$

Each new bond is generated by two summations of this type. Then the final result is the new partition function

$$Z=\sum_S \exp\Big[\beta'\sum_{i\nu}S_iS_{i+\nu}\Big] \tag{8}$$

defined on a lattice whose spacing is $\sqrt{2}$ in the original units.

This process has generated the transformation of parameters

$$\beta\to\beta'=2\tanh^{-1}\tanh^2\beta\ . \tag{9}$$

where β' is the effective value of β on the new lattice. This is a recursion formula. It is useful to analyze its behavior for small and large values of β. For small β,

$$\beta\to\beta'=2\beta^2+\cdots \tag{10}$$

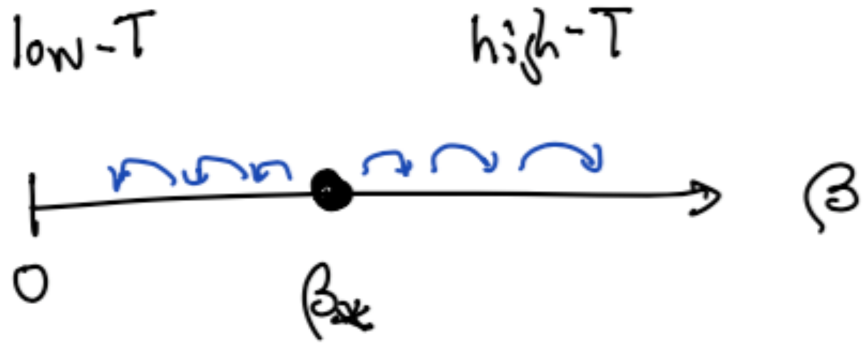
For large β, we can use the relation $\tanh\beta=1-2e^{-2\beta}+\cdots$ to write

$$\tanh^2\beta=(1-2e^{-2\beta})^2=(1-4e^{-2\beta})=1-2e^{-2(\beta+\log 2/2)}\ . \tag{11}$$

This gives the recursion

$$\beta \to \beta' = 2(\beta - \frac{1}{2}\log 2) + \cdots \tag{12}$$

We find that β decreases when β is small and increases when β is large.



(13)

By continuity, there must be a fixed point β_* at which

$$\beta_* = \beta'_* \ . \tag{14}$$

Under the recursion, smaller values of β flow toward $\beta = 0$, with the correlation length continually decreasing in units of the current lattice spacing. For small initial β, the effective β decreases, the correlation length becomse smaller in units of the current lattice spacing. Then we are in a high-temperature phase of the magnet. For large initial β, the successive β' values are larger and larger, indicating larger and more frozen magnetization. So all of these points are in the low-temperature phase. The fixed-point value β_* must correspond to the critical point β_c. The fact that the recursion formula at the value β_* gives back the same value β_* indicates that the physics of this point is scale-invariant.

This calculation gives an approximate calculation of T_c for the 2-dimensional Ising model, but we see that our approximations do not work very well. Numerically solving for the fixed point of the recursion formula, we find

$$\beta_* = 0.6094 \tag{15}$$

This value above differs from the exact value

$$\beta_c = \frac{1}{2}\log(1 + \sqrt{2}) = 0.4407 \ , \tag{16}$$

though it is certainly better than the value given by mean field theory, $\beta_c = 1/2d = 0.25$. We could systematically improve this approximation by constructing the recursion more exactly, using a larger set of effective interactions.

Within our current approximation, however, there is one more thing that we can learn. Let's study the recursion formula in the neighborhood of the fixed point. Values of β near β_* correspond to the critical region, the region near T_c where the correlation length is large.

I would like to linearize the recursion formula near the fixed point. Writing the recursion formula as

$$\tanh(\beta'/2) = \tanh^2 \beta \tag{17}$$

we can expand

$$\frac{1}{2}\frac{1}{\cosh^2(\beta'/2)}\ \Delta\beta' = 2\frac{1}{\cosh^2\beta}\tanh\beta\ \Delta\beta \tag{18}$$

and find

$$\frac{\Delta\beta'}{\Delta\beta} = 4\tanh\beta\frac{\cosh^2\beta'/2}{\cosh^2\beta}. \tag{19}$$

Setting $\beta = \beta_* + \Delta\beta$, $\beta' = \beta_* + \Delta\beta'$, we find

$$\Delta\beta' = \lambda\cdot\Delta\beta \qquad \lambda = 1.679 \tag{20}$$

The value of $\Delta\beta$ increases at each iteration, so the fixed point is *unstable*. After n iterations,

$$\Delta\beta^{(n)} = \lambda^n\ \Delta\beta \tag{21}$$

If the initial value of β is in the critical region, so that the initial $\Delta\beta$ is small, with $\beta < \beta_*$, the recursion will eventually take us to a value where $\Delta\beta$ is of order 1 and $\beta^{(n)}$ is significantly smaller than β_*. Then the system will be at high temperature and will have a correlation length comparable to the (current) lattice spacing. If we arrive at $\Delta\beta \sim 1$ in n steps, the lattice size will be

$$(\sqrt{2})^n \tag{22}$$

in units of the original lattice spacing. Then the correlation length in the original lattice units is

$$\xi = 2^{n/2} \tag{23}$$

or

$$\Delta\beta = (1/\lambda)^n \qquad \text{or} \qquad n = -\log\Delta\beta/\log\lambda\ . \tag{24}$$

This gives the relation

$$\xi = \exp[-(\log 2/2)\ (\log\Delta\beta/\log\lambda)]\ , \tag{25}$$

or

$$\xi = (\Delta\beta)^{-\log 2/2\log\lambda}\ . \tag{26}$$

Since $\Delta\beta \sim (T - T_c) \sim t$ we find

$$\xi \sim t^{-\nu} \qquad \text{with} \qquad \nu = 0.67 \tag{27}$$

This result should be compared to $\nu = 1/2$ in mean field theory and $\nu = 1$ in the exact solution.

The result of this calculation is not very impressive, but the principle of the calculation is very interesting. By integrating out a subset of the S_i, we have removed half of the degrees of freedom of the model, generating a new model with a lattice spacing a factor of $\sqrt{2}$ larger than the original one. This process generates a recursion formula for the parameters of the model. Within the approximation scheme, the prediction for the critical exponent ν is given by the properties of the recursion formula in the vicinity of the fixed point.

The exponent η is associated with the scaling behavior of the spin operator $S(x)$. It is possible to compute η by adding a term

$$\mathcal{H} \to \mathcal{H} - H \sum_i S_i \tag{28}$$

to the original Hamiltonian. It can be shown that this leads to a second unstable direction from the fixed point, whose instability allows us to compute η.

This turns out to be a powerful method for studying the critical region. Using an improved set of approximations, we can integrate out a fraction of the degrees of freedom of the model more accurately. The effective Hamiltonians at the various stages of the recursion will contain more interactions than the simple nearest neightbor coupling, but we can keep track of as many of these interactions as we wish. Let the coefficients of these terms be $\{c_i\}$, where c_1 is the coefficient of the nearest-neighbor coupling.

The discussion above suggests that:

1. The method of "integration out" generates a recursion formula in the space of the c_i.

2. This recursion relation has a fixed point $\{c_{i*}\}$. The value of β in the original model that evolve into this fixed point is the critical temperature β_c.

3. The fixed point is *unstable*. Models with an original value of $\beta < \beta_c$ will evolve away from the fixed point to the high-temperature phase, and models with an original value of $\beta > \beta_c$ will evolve away from the fixed point to the low-temperature phase.

4. The rate of instability of the fixed point gives a prediction for the critical exponent ν.

5. Adding a symmetry-breaking magnetic field to the original model, the fixed point acquires a second unstable direction. The rate of instability of the fixed point in this direction determines the critical exponent η.

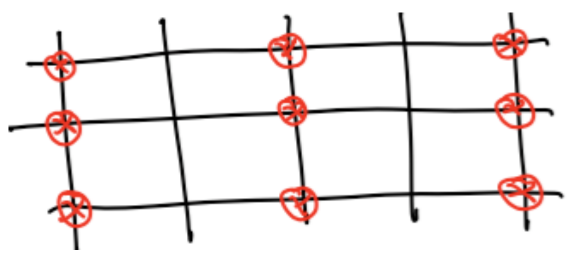
6. Once we have determined η and ν, we can obtain all of the other critical exponents using scaling arguments.

The idea of "integrating out", and this approach to the calculation of critical exponents, was developed independently by Kenneth Wilson and Leo Kadanoff in the 1960's.

The recursion formula that I have written here is a first example of a *renormalization group* (RG) transformation. In general, an RG transformation is a transformation on the parameters of a model that is generated by changing the underlying atomic length scale of the model by integrating out. Almost always, RG transformations takes place in a space of many parameters. The ideas of a fixed point and an unstable trajectory must be generalized to this multi-dimensional case. We will deal with this issue in an example specific to the theory of critical exponents in the next lecture.

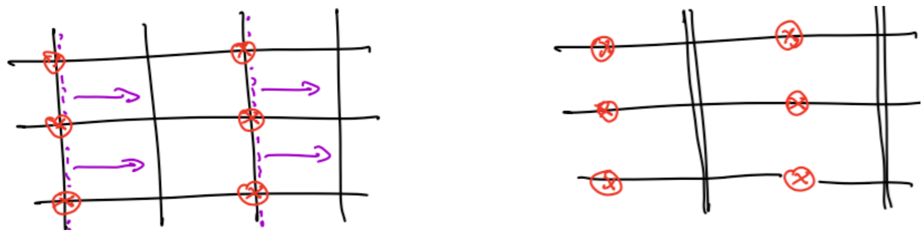
First, though, I would like to redo the above calculation using a less crude method, first proposed by Alexander Migdal and improved by Leo Kadanoff. This method is still approximate, but it takes better account of the various new interactions that are generated by the transformation, actually adapting the transformation to form a closed system for the recursion of parameters.

For the case of the 2-dimensional Ising model, Migdal chose to reduce the size of the lattice successively in the horizontal and then the vertical directions. We can reduce the horizontal sites by a factor of 2 by summing over columns of spins,



(29)

If we were to do this exactly, the sum over the horizontal bonds would be easy, but the sum over the vertical bonds would be more difficult, since these couple the neighboring rows. Migdal suggested the approximation of moving these vertical bonds out of the way, doubling the strength of the vertical interactions that are not summed over at this step,



(30)

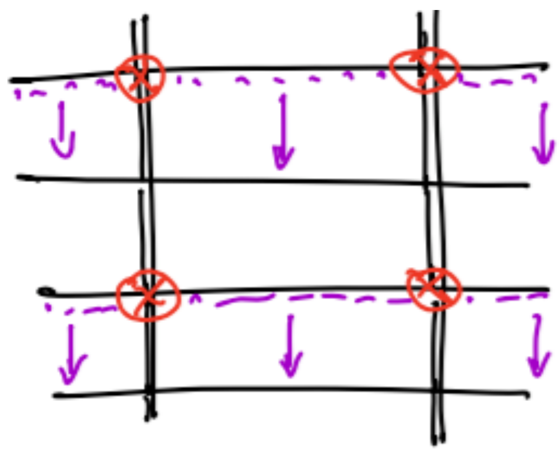
Starting from (3) with $z = \tanh \beta$ as before, the new vertical bonds have

$$\beta' = 2\beta \qquad \text{or} \qquad z' = \tanh 2\beta \tag{31}$$

The summation over the horizontal bonds then gives

$$\frac{1}{2}\sum_{S_0}(1 + zS_0S_1)(1 + zS_0S_2) = (1 + z^2 S_1S_2)\ . \tag{32}$$

Next, we thin the lattice by a factor of 2 in the vertical direction by summing over the spins in alternate rows, now pushing the horizontal bonds that make this summation awkward out the way.



(33)

The sums have the form

$$\frac{1}{2}\sum_{S_0}(1 + z'S_0S_1)(1 + z'S_0S_2) = (1 + (z')^2 S_1S_2) \tag{34}$$

After one full step, the vertical bonds have been transformed by

$$\beta^{(1)} = \tanh^{-1}(\tanh^2 2\beta) \tag{35}$$

and the horizontal bonds have been transformed by

$$\beta^{(1)} = 2\tanh^{-1}(\tanh^2 \beta) \tag{36}$$

The horizontal transformation is the same one as we saw in (9), except that the step in length scale is now a factor of 2 rather than $\sqrt{2}$. The recursion for the vertical bonds is equivalent. The two transformations have the same fixed point and the same rate of instability. The prediction for ξ is now

$$\xi = (\Delta\beta)^{-\log 2/\log\lambda} \ . \tag{37}$$

Since $\Delta\beta \sim (T - T_c) \sim t$ we find

$$\xi \sim t^{-\nu} \qquad \text{with} \qquad \nu = 1.34 \ . \tag{38}$$

Though I have explained the Migdal transformation for thinning the lattice by a factor of 2, we could use the same procedure to thin the lattice by a factor of $p = 3$, 4, etc. The horizontal transformation would be

$$\beta^{(1)} = p \tanh^{-1}(\tanh^p \beta) \ . \tag{39}$$

It is interesting to study this formula for an infinitestimal change of scale, $p = (1+\epsilon)$. Using

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2} \ , \tag{40}$$

we find in this limit

$$\begin{aligned} \beta^{(1)} &= (1+\epsilon)\tanh^{-1}(\tanh\beta(1+\epsilon\log\tanh\beta)) \\ &= (1+\epsilon)\tanh^{-1}(\tanh\beta) + \frac{\epsilon}{1-\tanh^2\beta}\tanh\beta\log\tanh\beta \end{aligned} \tag{41}$$

Then

$$\beta^{(1)} - \beta = \epsilon\Big(\beta + \frac{\sinh\beta}{\cosh\beta}\cosh^2\beta\log\tanh\beta\Big) \ . \tag{42}$$

This gives a differential equation for the effective β as a function of the current length scale ℓ

$$\ell\frac{d}{d\ell}\beta(\ell) = \beta + \frac{1}{2}\sinh 2\beta\log\tanh\beta \ . \tag{43}$$

The fixed point occurs where the right-hand side of this equation vanishes. It is quite remarkable that, if we put in the exact value of the critical temperature

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}) \tag{44}$$

then

$$\sinh 2\beta_c = 1 \ , \quad \cosh 2\beta_c = \sqrt{2}, \qquad \log\tanh\beta_c = -\log(1+\sqrt{2}) \ . \tag{45}$$

Plugging these expressions into (43), we see that the fixed point is exactly at β_c.

Now expand about the fixed point

$$\ell\frac{d}{d\ell}(\beta-\beta_c) = (\beta-\beta_c)\left[1+\frac{1}{2}\sinh 2\beta_c\frac{1}{\tanh\beta_c\cosh^2\beta_c}+\cosh 2\beta_c\log\tanh\beta_c\right] \tag{46}$$

The second term in backets equals 1. Then we find

$$\ell\frac{d}{d\ell}(\beta-\beta_c) = (\beta-\beta_c)\left[2-\sqrt{2}\log(1+\sqrt{2})\right] \tag{47}$$

or

$$\ell\frac{d}{d\ell}(\beta-\beta_c) = 1.459\ (\beta-\beta_c)\ . \tag{48}$$

The solution of this equation is the power law

$$\Delta\beta(\ell) = \ell^{1.459}\ (\Delta\beta) \tag{49}$$

We can take the final value of ℓ such that $\Delta\beta(\ell)\sim 1$. Then the original value of $\Delta\beta$ satisfies

$$1 = \xi^{0.754}\ (\Delta\beta)\ , \tag{50}$$

which implies

$$\xi \sim (T-T_c)^{1/1.459}\ . \tag{51}$$

Then the prediction for ν is

$$\nu = 0.686 \tag{52}$$

This should be compared to $\nu = 1$ in the exact solution.

This example gives a concrete idea of what we would like to achieve in a more exact analysis. We would like to integrate out degrees of freedom within a large model space with parameters $\{c_i\}$. Then we will find an evolution equation for the $\{c_i\}$ of the form

$$\ell\frac{d}{d\ell}c_i(\ell) = \mathcal{F}_i[\{c_j\}] \tag{53}$$

The $c_i(\ell)$ are the effective parameters of the model after integrating out all degrees of freedom from the original underlying length scale a to the final underlying length scale ℓ. The fixed points of this transformation will give the value of T_c and the values of the critical exponents. In the next two lectures, I will present some examples of this analysis that will bring us closer to the practical calculation of critical exponents.