Oct. 17

Physics 212 – Statistical Mechanics

Landau Theory of Phase Transitions

Through the study of explicit models of magnetism, we have learned many facts about the phase diagrams of systems with spontaneous symmetry breaking. Now it is time to step back and try to create a global picture of the behavior of these systems. This picture will be frankly phenomenological. We will make some assumptions, which will be questioned later in the course. But I hope you will find this picture straightforward to construct and easy to analyze. It will give us quantitative predictions for the form of the phase diagram in the Ising model and in more complex systems.

This picture is due to Lev Landau. It this course, we will discuss the basic applications of the Landau theory. More sophisticated applications that go deeper into solid-state physics are given in the Landau and Lifshitz *Statistical Physics* volume.

In this lecture, I will present the Landau description of the Ising model. In the previous lectures, we computed the free energy F and tried to find the thermodynamic state by minimize our approximate expressions for F. In the Landau theory, the starting point is the Gibbs free energy

$$G = F + HM \tag{1}$$

which is a function of T and M and satisfies

$$\left.\frac{\partial G}{\partial M}\right|_T = H \ . \tag{2}$$

Roughly, G represents the energy-entropy balance in the presence of an external H field that maintains the expectation value of the magnetization at the value M, By virtue of the above equation, the extrema of $G(M)$ correspond to consistent thermodynamic states with $H = 0$. Thus, they give the possible values of the spontaneous magnetizatio in zero field. The actual thermodynamic state for given T is given by the *minimum* of $G(T, M)$. Symmetry might require that there are several degenerate ground states of $G(T, M)$, corresponding to coexisting thermodynamic states.

In the fundamental description of the Ising model, the spin variables S_i take the quantized values ± 1. However, as the spins fluctuate from site to site, the coarse-grained average of of spins over a small volume will generally take a smaller value. I would like now to write a phenomenological description of the magnet, using this locally averaged spin as a *local magnetization* $m(x)$ and treating it as a variable on a continuous background space. Thus, I will write the total magnetization as

$$M = \int d^d x \; m(x) \; . \tag{3}$$

Assume that the magnetization is due only to short-range interactions. Then the Gibbs free energy as a function of $m(x)$ will take the form

$$G = \int d^d x \; g[m(x)] \; . \tag{4}$$

were $g[m(x)]$ a functional depending on $m(x)$.

In the region near $T = T_c$, the local magnetization $m(x)$ will be small, and it will be reasonable to expand the function $g[m(x)]$ in powers of $m(x)$, keeping only the leading terms. I propose

$$G = \int d^d x \Big\{ \frac{1}{2}\rho(\vec{\nabla} m)^2 + \frac{1}{2} A m^2 + \frac{1}{4} B m^4 \Big\} \tag{5}$$

The first term expresses the fact that it costs free energy to have $m(x)$ differ from one point to another in equilibrium. The A and B terms represent the relative free energies of different values of $m(x)$. I claim that we should choose $A(T)$ to vanish as $T = T_c$. You will see the virtue of this in a moment. With these assumptions, we can write G in the form

$$G = \int d^d x \Big\{ \frac{1}{2}(\vec{\nabla} m)^2 + \frac{1}{2} a(T - T_c) m^2 + \frac{1}{4} b m^4 \Big\} \tag{6}$$

with $a, b > 0$, and I have absorbed ρ into the normalization of $m(x)$. I emphasize again that this expression is valid only in the vicinity of $T = T_c > 0$. This is Landau's proposal.

I should emphasize that the approximation $A(T) = a(T - T_c)$ is made as a phenomenological description. Ideally, A and B should be computed from an underlying theory.

If we would like to discuss the influence of the external field H explicitly, we can go back to F. From the expression (6) for G, F takes the form

$$F = \int d^d x \Big\{ \frac{1}{2}(\vec{\nabla} m)^2 + \frac{1}{2} a(T - T_c) m^2 + \frac{1}{4} b m^4 - H m \Big\} \Big|_{\min} \; , \tag{7}$$

where the expression is to be *minimized* with respect to the function $m(x)$.

Let's first study the system at $H = 0$. We consider constant values $m(x) = m$. Then the minimzation equation for G is

$$\frac{\partial G}{\partial m} = V \cdot (a(T - T_c) m + b m^3) \tag{8}$$

where V is the volume of the system. For $T > T_c$, there is only one solution

$$m = 0 \tag{9}$$

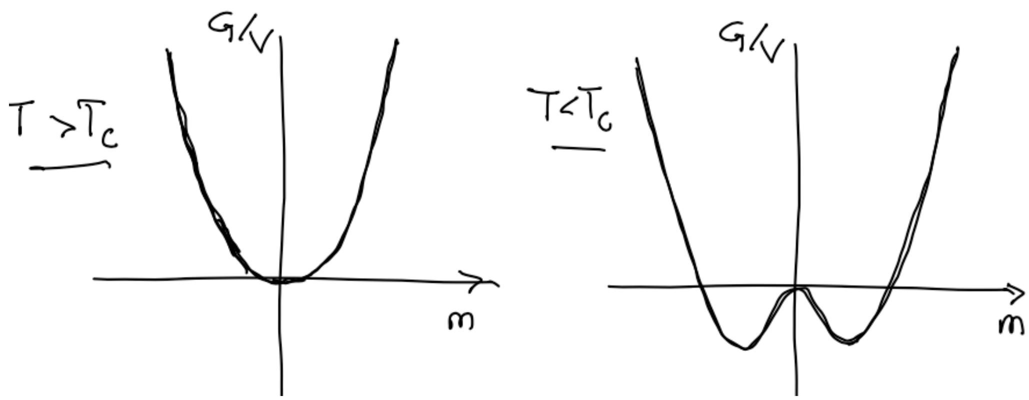
For $T < T_c$, the equation is

$$a(T_c - T)m = bm^3 \tag{10}$$

and there are 3 solutions

$$m = 0 \qquad m = \pm\left[\frac{a(T_c - T)}{b}\right]^{1/2} \tag{11}$$

We can see the origin of this by plotting $G[m]/V$



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For $T > T_c$, G has a single global minimum at $m = 0$. For $T < T_c$, the state $m = 0$ is unstable, and there are two stable minima at symmetrical nonzero values. Thus, the Landau expression for the Gibbs free energy produces the by now familiar results

$$T > T_c \ : \quad M = 0 \qquad T < T_c \ : \quad M = \pm C \ (T_c - T)^{1/2} \ . \tag{13}$$

For $H > 0$, the quantity to be minimized in (7)

$$\frac{1}{2}a(T - T_c)m^2 + \frac{1}{4}bm^4 - Hm \ . \tag{14}$$

For $T > T_c$, assuming a small deviation from the $H = 0$ equilibrium state, the minimization equation is, for $T > T_c$,

$$0 = a(T - T_c)m + \mathcal{O}(m^3) - H \ , \tag{15}$$

so that

$$m = \frac{H}{a(T - T_c)} + \cdots \tag{16}$$

The spin susceptibility at $H = 0$ is then

$$\chi = \frac{\partial M}{\partial H} \sim \frac{1}{T - T_c} \ . \tag{17}$$

More generally, the minimization equation is

$$0 = a(T - T_c)m + bm^3 - H \ , \tag{18}$$

Then, for $T = T_c$

$$m = \left(\frac{H}{b}\right)^{1/3} \qquad \text{or} \qquad M \sim H^{1/3} \tag{19}$$

To study the case $T < T_c$, we expand

$$m = m_o + \Delta m \qquad m_o = \left[\frac{a(T_c - T)}{b}\right]^{1/2} \tag{20}$$

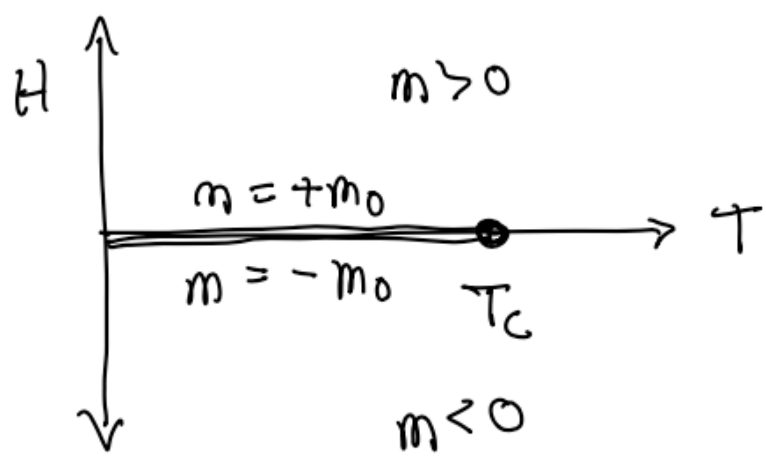
Then the equation (18) becomes

$$\begin{aligned} 0 &= -a(T_c - T)(m_0 + \Delta m) + b(m^3 + 3m^2\Delta m + \cdots) \\ &= (3bm_0^2 - a(T_c - T))\Delta m - H + \cdots \\ &= 2bm_0^2 \Delta m - H \end{aligned} \tag{21}$$

so m is slightly enhanced from its $H = 0$ value and

$$\chi = \frac{\partial M}{\partial H} = V\frac{1}{2bm_0^2} \sim \frac{1}{T_c - T} \tag{22}$$

In all, we find that the Landau theory gives us back the phase diagram



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in agreement with the structure that we found in mean field theory.

The value of G or F at zero external field is

$$\begin{aligned} F/V &= \frac{1}{2}a(T - T_c)m_0^2 + \frac{1}{4}bm_0^4 \\ &= \begin{cases} 0 & T > T_c \\ -(a^2/4b)(T_c - T)^2 & T < T_c \end{cases} \end{aligned} \tag{24}$$

This implies

$$C = \frac{\partial^2 F}{\partial T^2} = \begin{cases} 0 & T > T_c \\ V \cdot (a^2/2b) & T < T_c \end{cases} \tag{25}$$

In all, we see that Landau theory gives in an easy way the same predictions as mean field theory for the nonanalytic behaviors of the thermodynamic functions in the region near T_c:

$$\begin{aligned} \text{at } H = 0,\ T \to T_c\ : & \qquad M \sim (T_c - T)^{1/2}\ ,\ \chi \sim |T - T_c|^{-1} \\ \text{at } T = T_c\ : & \qquad M \sim H^{1/3}\ ,\ C \text{ discontinuous} \end{aligned} \tag{26}$$

Landau theory also allows us to explore the behavior of correlation functions. Within Landau theory, we represent

$$\langle S_I S_J \rangle \to \langle m(x) m(y) \rangle \tag{27}$$

where the expectation value should be computed using the probability distribution

$$\exp[-\beta G[m]] \tag{28}$$

This raises a mathematical problem: We need a systematic way to sum — or, rather, integrate — this distribution over all possible values of the magnetization $m(x)$. Let's write this integral — a *functional integral* — as

$$\int \mathcal{D}m(x)\ e^{-\beta G[m]} \tag{29}$$

The difficulty of evaluating this integral depends on the form of the functional $G[m(x)]$. Let's first consider the case $T > T_c$. Here it is a good approximation to write

$$G[m] = \int d^d x \left[\frac{1}{2} (\vec{\nabla} m)^2 + \frac{1}{2} A m^2 \right] . \tag{30}$$

This expression is quadratic in the variable $m(x)$ and so we have a Gaussian integral. This is a very lucky situation. Gaussian integrals are exceptionally easy to evaluate. Let's pause here and review the basic equations.

Begin with 1 variable. You all know Liouville's result

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi} \tag{31}$$

(Kelvin: "A mathematician is one to whom that is as obvious as that twice two makes four is to you. Liouville was a mathematician.") By a simple rescaling,

$$\int_{-\infty}^{\infty} dx e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}} \ . \tag{32}$$

Now let's compute the moments of this as a probability distribution.

$$\langle x^n \rangle = \int dx e^{-ax^2/2} x^n \ / \ \int dx e^{-ax^2/2} \tag{33}$$

Obviously, $\langle 1 \rangle = 1$. Also, by symmetry, $\langle x^n \rangle = 0$ if n is odd.

For n even, we can construct the integrals that we need by differentiating with respect to a,

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^2 &= -2\frac{\partial}{\partial a} \int_{-\infty}^{\infty} dx e^{-ax^2/2} = (-2)(-\frac{1}{2})\frac{1}{a} \cdot \sqrt{\frac{2\pi}{a}} \\ \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^4 &= -2\frac{\partial}{\partial a} \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^2 = (2 \cdot \frac{3}{2})(2 \cdot \frac{1}{2})\left(\frac{1}{a}\right)^2 \cdot \sqrt{\frac{2\pi}{a}} \end{aligned} \tag{34}$$

Then we see that

$$\langle 1 \rangle = 1 \ , \qquad \left\langle x^2 \right\rangle = \frac{1}{a} \qquad \left\langle x^4 \right\rangle = \frac{3 \cdot 1}{a^2} \qquad \left\langle x^6 \right\rangle = \frac{5 \cdot 3 \cdot 1}{a^3} \tag{35}$$

The general case is

$$\left\langle x^{2n} \right\rangle = \frac{(2n-1) \cdot (2n-3) \cdot 3 \cdot 1}{a^n} \ . \tag{36}$$

This set of results has an interesting pictorial interpretation. Let the "contraction" of a pair of x variables be

$$\overline{x \ x} = \left\langle x^2 \right\rangle = \frac{1}{a} \tag{37}$$

Then we can evaluate $\langle x^4 \rangle$ as a sum of contractions

$$\begin{aligned} \left\langle x^4 \right\rangle &= \langle xxxx \rangle \\ &= \langle xxxx \rangle + \langle xxxx \rangle + \langle xxxx \rangle \\ &= 3 \cdot (1/a)^3 \end{aligned} \tag{38}$$

If n is odd, we have an x left over, and this gives $\langle x^n \rangle \sim \langle x \rangle = 0$. So the expectation values vanish by this rule for all odd values of n. The combinatorics is such that the cases of n even reproduce the results above

$$\left\langle x^{2n} \right\rangle = (2n-1)(2n-3) \cdots 1 \cdot \frac{1}{a^n} \tag{39}$$

The general rule, called "Wick's theorem", is

> The expectation value of a product of Gaussian random variables is equal to the sum of all possible contractions.

Let's generalize this to the case of 2 variables. If the Gaussian integral is

$$\int_{-\infty}^{\infty} dx e^{-a_1 x_1^2/2 - a_2 x_2^2/2} = \sqrt{\frac{2\pi}{a_1}}\sqrt{\frac{2\pi}{a_2}} \tag{40}$$

we can evaluate

$$\begin{aligned} \overbracket{x_1 x_1} = \left\langle x_1^2 \right\rangle = \frac{1}{a_1} \qquad \overbracket{x_2 x_2} = \left\langle x_2^2 \right\rangle = \frac{1}{a_2} \\ \overbracket{x_1 x_2} = \langle x_1 x_2 \rangle = 0 \ . \end{aligned} \tag{41}$$

You can see that expectation values of products of x_1 and x_2 can then be evaluated by contractions using (41).

More generally, we can consider a Gaussian integral with a general real quadratic form in the exponent,

$$\int_{-\infty}^{\infty} dx \ \exp[-A_{ab} x_a x_b/2] \tag{42}$$

with $a, b = 1, 2$. If A_{ab} were diagonal, we could evaluate any expectation value of the x_a using the contractions in (41). On the other hand, since A is real symmetric, we can always diagonalize it by a change of basis,

$$A = R \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \ . \tag{43}$$

In the new basis (y_1, y_2),

$$\begin{aligned} \overbracket{y_1 y_1} = \left\langle y_1^2 \right\rangle = \frac{1}{\lambda_1} \qquad \overbracket{y_2 x_2} = \left\langle y_2^2 \right\rangle = \frac{1}{\lambda_2} \\ \overbracket{y_1 y_2} = \langle y_1 y_2 \rangle = 0 \ . \end{aligned} \tag{44}$$

To transform back to the orignal basis, note that

$$R \begin{pmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{pmatrix} R = A^{-1} \ . \tag{45}$$

In this basis, the contractions are given by

$$\overbracket{x_a x_b} = \langle x_a x_b \rangle = (A^{-1})_{ab} \tag{46}$$

and all Gaussian expectation values of x_a, x_b can be evaluated by Wick's theorem using this contraction. This result generalizes if there are more than two integration variables. In fact, we have now proved Wick's theorem for a Gaussian integral over any number of variables.

We can now go back to the problem at hand, the evaluation of expectation values of $m(x)$. Here there are an infinite number of random variables, so in principle we should be careful in defining the integral. I claim that, with any reasonable regularization, the above result should still hold. There is a straightforward choice for the matrix A_{ab}. In this case, the matrix becomes a linear operator on a space of functions. The Gaussian probability distribution is

$$\exp\Big[-\frac{\beta}{2}\int d^dx \; ((\vec{\nabla} m)^2 + Am^2)\Big] \tag{47}$$

By integration by parts, we can put the exponent into the form

$$-\frac{\beta}{2}\int d^dx \; m(x)(-\nabla^2 + A)m(x) \; . \tag{48}$$

Here we see $\mathcal{O} = (-\nabla^2 + A(T))$ is a linear operator acting on the variables $m(x)$ The inverse of this operator is the Green's function $G(x, y)$ that gives the solution to the differential equation

$$\beta(-\nabla_x^2 + A) \; G(x, y) = \delta^d(x - y) \tag{49}$$

Thus,

$$\langle m(x)m(y)\rangle = G(x, y) \tag{50}$$

and this will give the contraction from which we can compute any expectation value of variables $m(x)$ using Wick's theorem.

If you would like more details of the definition and regularization of functional integrals, you might begin with Chapter 9 of my quantum field theory textbook (Peskin & Schroeder). For the rest of this course, the results quoted above will be sufficient.

From the general discussion of Gaussian integrals, we see that, for $T > T_c$,

$$\langle m(x)m(y)\rangle = G(x, y) \tag{51}$$

up to corrections due to the quartic term in the Landa free energy. IN the following, I will assume that those corrections are small, which it correct well away from $T = T_c$. To find the behavior of the correlation function (51), we need to solve the equation (49) for $G(x, y)$ for the relevant value of the space dimension d. We want to find solutions to this equation that remain finite as $|x - y| \to \infty$.

In 1 dimension, the equation is

$$\beta\Big(-\frac{\partial^2}{\partial x^2} + A(T)\Big)G(x, y) = \delta(x - y) \tag{52}$$

The delta function on the right-hand side of the equation implies a discontinuity in the first derivative of $G(x, y)$ at $x = y$. The solution that remains finite as $|x - y| \to \infty$

is proportional to $\exp[-A^{1/2}|x-y|]$ in this limit. The solution with the correct discontinuity is

$$G(x,y) = \frac{T}{2\sqrt{A}} e^{-\sqrt{A}|x-y|} \ . \tag{53}$$

In three dimensions, for $A = 0$, the solution is the Coulomb potential

$$G(x,y) = \frac{T}{4\pi} \frac{1}{|x-y|} \ . \tag{54}$$

For $A > 0$, the equation is that of a Yukawa potential

$$(-\nabla^2 + \mu^2) V(x) = \delta(\vec{x}) \tag{55}$$

Then the solution for general positive A is

$$G(x,y) = \frac{T}{4\pi} \frac{1}{|x-y|} e^{-\sqrt{A}|x-y|} \ . \tag{56}$$

In 2 dimensions, the solution of the equation for $A = 0$ is the 2-dimensional electrostatic potential, which is logarithmic

$$V(x) = -\frac{1}{2\pi} \log |x-y| \tag{57}$$

so the Green's fuction that gives the spin-spin correlation function is

$$G(x,y) = -\frac{T}{2\pi} \log \sqrt{A}|x-y| \ . \tag{58}$$

In a general dimensionality, for $A = 0$, one can use Gauss' law (or, for the power law, dimensional analysis) to show that

$$G(x,y) = \frac{T}{\mathcal{A}_d} \frac{1}{|x-y|^{d-2}} \tag{59}$$

where $\mathcal{A}_d$ is the area of the unit sphere in d dimensions. It can be shown (see the appendix) that, for $A > 0$, the Green's function has an exponential fall off

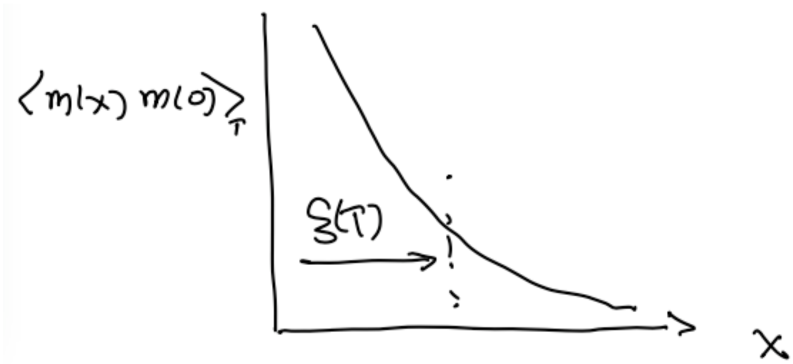
$$G(x,y) \sim \frac{T}{|x-y|^{(d-1)/2}} \exp[-\sqrt{A}|x-y|] \ . \tag{60}$$

Almost always in the course, the above expressions for $G(x,y)$ will be sufficient.

Always, for $A(T) > 0$, $T > T_c$, the correlation function has an exponential fall-off of the form

$$G(x-y) \sim \exp[-|x-y|/\xi(T)] \ , \tag{61}$$

graphically,



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The correlation length $\xi(T)$ is given by

$$\xi(T) = [a(T - T_c)]^{1/2} \tag{63}$$

Notice that this function goes to infinity as $T \to T_c$.

In the transfer matrix picture, there is an interesting interpretation of this result. The field $m(x)$ is a boson field whose wave-like modes of wavenumber k have the frequencies

$$\omega_p = [k^2 + a(T - T_c)]^{1/2} \tag{64}$$

This is a relativistic spectrum similar to the one that we saw in the 2-dimensional Ising model, with the particle mass equal to

$$m(T) = [a(T - T_c)] \; . \tag{65}$$

In this picture, the correlation length is equal to the boson Compton wavelength. Notice that $m(T) \to 0$ as $T \to T_c$, just as we saw for the fermion mass in the Ising model, leading to an infinite correlation length at T_c.

In the Landau theory, then, there are are two additional scaling laws in the region $T \sim T_c$,

$$\begin{aligned} \text{at } H = 0,\ T \to T_c \ : \qquad & \xi(T) \sim 1/(T_c - T)^{1/2} \ , \\ \text{at } H = 0,\ T = T_c \ : \qquad & \langle m(x) m(y) \rangle \sim 1/|x - y|^{d-2} \end{aligned} \tag{66}$$

In Landau theory, it is also easy to work out the behavior of correlation functions for $T < T_c$. To do this, expand the integral over $G[m(x)]$ about the minimum of G at (for definiteness) positive $\langle m \rangle$. Thus, write

$$m(x) = m_0 + \Delta m(x) \qquad m_0 = \left[\frac{a(T_c - T)}{b} \right]^{1/2} \tag{67}$$

and expand

$$
\begin{aligned}
&\frac{1}{2}(\vec{\nabla} m)^2 + \frac{1}{2}a(T-T_c)m^2 + \frac{1}{4}bm^4 \\
&= \frac{1}{2}(\vec{\nabla}\Delta m)^2 + \frac{1}{2}(a(T-T_c)m_0^2 + 2m_0\Delta m + \Delta m^2) \\
&\qquad + \frac{1}{4}(m_0^4 + 4m_0^3\Delta m + 6m_0^2\Delta m^2 + \cdots) \\
&= c(m_0) + \frac{1}{2}(\vec{\nabla}\Delta m)^2 + (a(T-T_c)+b)m_0\Delta m + \frac{1}{2}(-2a(T_c-T) + 3m_0^2)\Delta m^2 + \cdots
\end{aligned}
\tag{68}
$$

The constant term $c(m_0)$ gives an overall factor in Z that does not contribute to correlation functions. The term proportional to $(\Delta m)^1$ cancels due to the equation (67) for m_0. Putting in the explicit value of m_0, the expression collapses to

$$
c(m_0) + \frac{1}{2}(\vec{\nabla}\Delta m)^2 + \frac{1}{2}(2a(T_c - T))\Delta m^2 + \cdots \ . \tag{69}
$$

Then the probability weight of a given configuration $\Delta m(x)$ is

$$
\exp\Big[-\beta \int d^dx \ \frac{1}{2}\Delta m(x)\Big\{-\nabla^2 + 2a(T_c - T)\Big\}\Delta m(x) \ . \tag{70}
$$

Following the same logic as above, this weight leads to an expectation value

$$
\langle \Delta m(x)\Delta m(y)\rangle = \overline{G}(x,y) \sim \exp[-|x-y|/\overline{\xi}(T)] \tag{71}
$$

with the correlation length

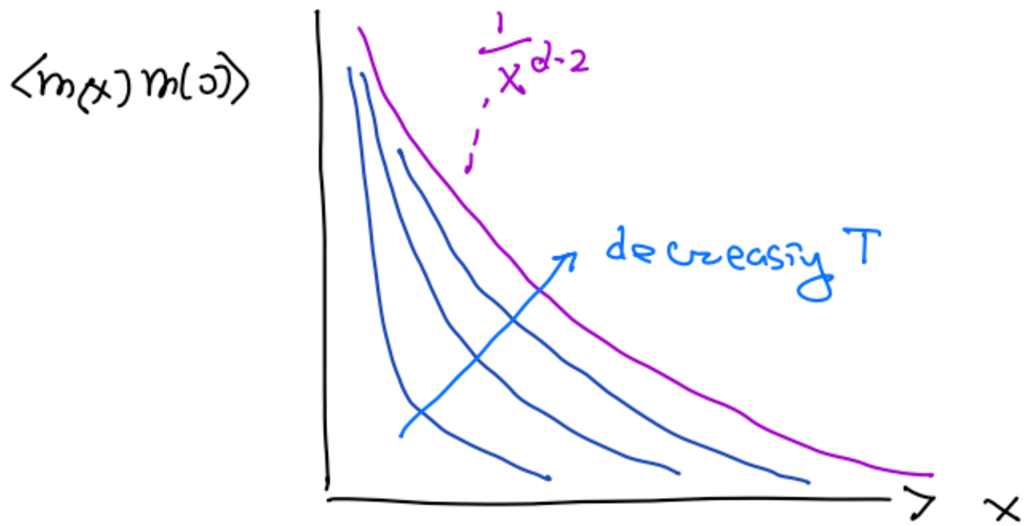
$$
\overline{\xi}(T) = [2a(T_c - T)]^{1/2} \tag{72}
$$

Since $\langle \Delta m(x)\rangle = 0$, the full correlation function of $m(x)$ has the value

$$
\langle m(x)m(y)\rangle = m_0^2 + \overline{G}(x,y) \ . \tag{73}
$$

I would now like to graph the behavior of the spin-spin correlation function according to Landau theory across the whole range of temperatures. For $T > T_c$, correlation function tends to zero as $|x-y| \to \infty$ while the correlation length increases as T

decreases:



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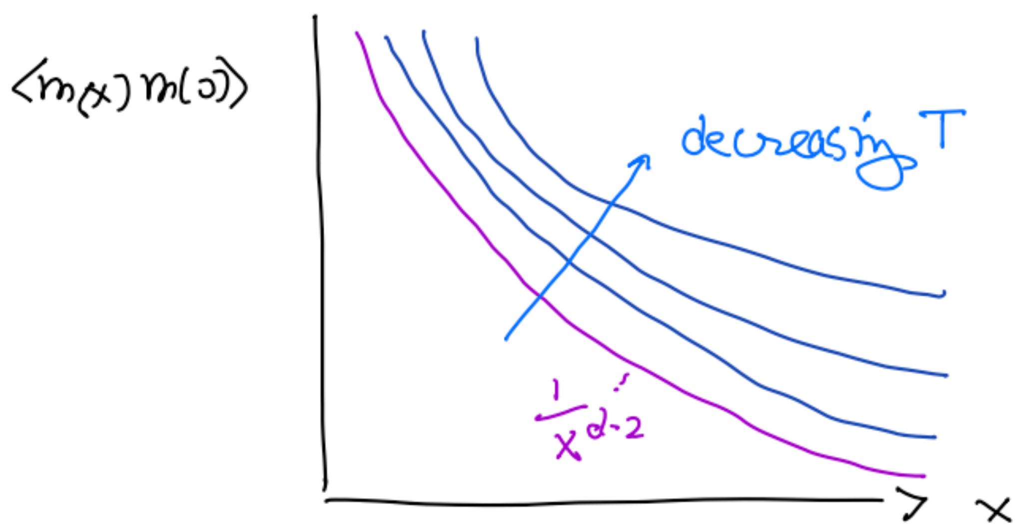
For $T < T_c$, the correlation length decreases as T decreases, but the asymptotic value of the correlation function, $m_0^2(T)$, increases T decreases.



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The two diagrams fit together across the curve for $T = T_c$, which is a pure power law decay.

Notice that $\langle m(x)m(0)\rangle$ is monotonically decreasing for any x as a function of increasing T, as it should be. Be sure that you understand how this is compatible with the decrease of the correlation length with T in the low temperature phase.

Here is some extra material for those who are curious about it:

For general d, the Green's function $G(x,y)$ is given as the solution to the equation

$$\beta(-\nabla^2 + m^2)G(x-y) = \delta^{(d)}(x-y) \ . \tag{76}$$

with $m^2 = A$. It is straightforward to solve this equation in any d. Fourier transform the equation to find

$$(k^2 + A)\tilde{G}(k) = T \tag{77}$$

Then

$$\tilde{G}(k) = \frac{T}{k^2 + m^2} \tag{78}$$

Return this to real space by computing the integral

$$G(x) = \int \frac{d^dk}{(2\pi)^d} \ e^{-i\vec{k}\cdot\vec{x}} \frac{T}{k^2 + m^2} \tag{79}$$

By symmetry, we can choose the z axis parallel to $\vec{k}$. Then the measure will be rewritten

$$d^dk = dkk^{d-1}d\theta(\sin\theta)^{d-2}\mathcal{A}_{d-1} \tag{80}$$

where $\mathcal{A}_{d-1}$ is the area of the unit sphere in $(d-1)$ dimensions. We then need to do the integral

$$\int_0^\pi d\theta \sin^{d-2}\theta \ e^{ikx\cos\theta} = \int_0^\pi d\theta \sin^{d-2}\theta \ \cos(kx\cos\theta) \ , \tag{81}$$

since $\cos(\pi - \theta) = -\cos\theta$. We are now in the realm of Bessel functions,

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_0^\pi d\theta \ \sin^{2\nu}\theta \ \cos(z\cos\theta) \ , \tag{82}$$

We have now done the θ integral for k real and > 0,

$$G(x) = \int_{-\infty}^\infty \frac{dkk^{d-1}}{(2\pi)^d}\mathcal{A}_{d-1}\frac{T}{k^2 + m^2}\Big[\frac{\sqrt{\pi}\Gamma((d-1)/2)}{(kx/2)^{d-2)/2}} \ J_{(d-2)/2}(kx)\Big] \tag{83}$$

I claim that this can be written

$$G(x) = \frac{1}{2}\int_{-\infty}^\infty \frac{dkk^{d-1}}{(2\pi)^d} \ \mathcal{A}_{d-1} \ \frac{T}{k^2 + m^2} \ \frac{\sqrt{\pi}\Gamma((d-1)/2)}{(kx/2)^{d-2)/2}} \ H^{(1)}_{(d-2)/2}(kx) \ , \tag{84}$$

where $H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z)$, $H^{(2)}_\nu(z) = J_\nu(z) - iY_\nu(z)$. To understand this, note that $H^{(1)}(z)$ has the analytic continuation

$$H^{(1)}_\nu(ze^{i\pi}) = -e^{-\nu\pi i}H^{(2)}(z) \ , \tag{85}$$

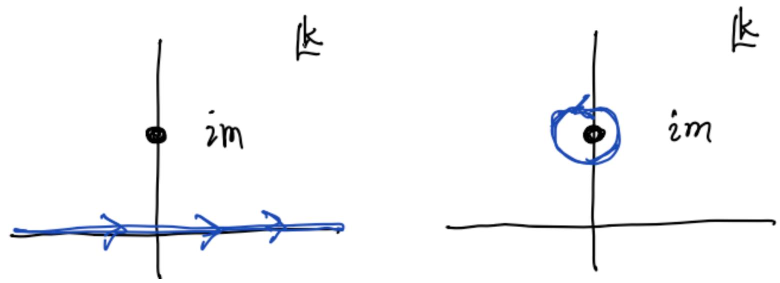
so the J_ν pieces add between k and $-k$ while the Y_ν pieces cancel. Using the formula for the area of the unit sphere

$$A_d = 2\pi^{d/2}/\Gamma(d/2) \tag{86}$$

and writing $w = kx$, we can simplify this to

$$G(x) = \frac{1}{2}\frac{1}{x^{d-2}}\int_{-\infty}^{\infty}\frac{dw\ w^{d-1}}{(2\pi)^d}\ \frac{T}{w^2+(mx)^2}\ \frac{(2\pi)^{d/2}}{w^{(d-2)/2}}\ H^{(1)}_{(d-2)/2}(w)\ . \tag{87}$$

The integrand of (84) is analytic in the upper half k plane, except for the pole at $k = im$,



(88)

The analytic continuation of $H^{(1)}_\nu(z)$ in this region is

$$H^{(1)}_\nu(e^{i\pi/2}w) = (-i)\frac{2}{\pi}e^{-\nu\pi i/2}K_\nu(w) \tag{89}$$

where $K_\nu(w)$ is the modified Bessel function that falls off exponentially as $w \to \infty$. So the integral (84) is equal to the contribution of the pole,

$$G(x) = \frac{1}{2}\frac{1}{x^{d-2}}\cdot(2\pi i)\frac{T}{2imx}\frac{(imx)^{d-1}}{(2\pi)^d}\ \frac{(2\pi)^{d/2}}{(imx)^{(d-2)/2}}\ (\frac{-2i}{\pi})\ (-i)^{(d-2)/2}\ K_{(d-2)/2}(mx) \tag{90}$$

or, finally,

$$G(x) = \frac{T}{2\pi}\left(\frac{m}{2\pi x}\right)^{(d-2)/2} K_{(d-2)/2}(mx) \tag{91}$$

Using the asymptotic limits

$$K_\nu(z) = \frac{1}{2}\Gamma(\nu)(z/2)^{-\nu} + \cdots \ \text{as } z \to 0 \tag{92}$$

and

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z} + \cdots \ \text{as } z \to \infty \tag{93}$$

you should be able to verify the results for $G(x)$ quoted above.