Oct. 19

Physics 212 – Statistical Mechanics

Goldstone Bosons

In the previous lecture, I discussed the Landau theory that describes the phase transition in the Ising model. In this lecture, I will generalize that description to a wide variety of other phase transitions.

One of the key ideas of Landau theory is that we coarse-grain average the order parameter and turn it into a field on a continuum background. If there is only one order parameter (equivalently, if the ordering can be described by a single field) that system would be described by exactly the same Landau theory that we set up in the previous lecture. On the other hand, if we have a system with several order parameters, or a multicomponent order parameter such as a vector spin $\vec{S}_i$, we need to make the appropriate generalization of the formulae we write previously. This generalization should use the same principles — keeping only the terms with the lowest relevant powers of the order parameter, considered as a field on a continuous space.

What systems are described by a 1-field Landau theory? Certainly, the Ising model of a magnet has this description. In the problem set, we studied an Ising antiferromagnet on a cubic lattice, for which the 1-component order parameter was the staggered magnetization

$$M_s = \sum_j (-1)^j S_j \ , \tag{1}$$

where $(-1)^j$ is $+1$ on even sites of a lattice and -1 on odd sites. But there are other examples. For the liquid-gas critical point, the order parameter is a 1-component field, the density. Similarly, there are 2-component liquid mixtures that separate into 2 phases at a critical point. Here, the order paramter is the mole fraction of component A as opposed to component B. Another example is a system called β-brass. This is a Cu–Zn alloy with a random arrangement of Cu and Zn atoms in the high temperature phase, but with an arrangement with Cu and Zn on alternate sites at low temperature. All of these systems are described within Landau theory by the model that we studied in the previous lecture, an effective Gibbs free energy with one field $m(x)$. This would seem to indicate that these systems have some common properties near the critical point. We will see later in the course that this is realized in a unexpectedly strong way.

For an XY magnet, the basic model has a spin degree of freedom S_i with 2 components. If we coarse-grain and turn this into a continuum field, that would be a

2-component field

$$\vec{m}(x) = (m^1, m^2) \tag{2}$$

with a Gibbs free energy invariant under an $SO(2)$ symmetry that rotates the spin variables. The Landau form of the Gibbs free energy is

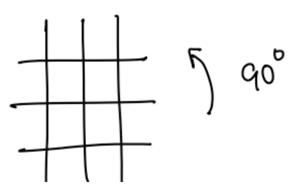
$$G = \int d^dx \Big\{ \frac{1}{2} (\sum_a (\vec{\nabla} m^a)^2 + \frac{1}{2} a(T - T_c) \sum_a (m^a)^2 + \frac{1}{4} b (\sum_a (m^a)^2)^2 \Big\} \tag{3}$$

where the index a is summed over the two components of $\vec{m}$. It turns out that the phase transitions to superfluidity and superconductivity are also described by this Landau theory. I will explain this in detail in next week's lectures.

This is not the only possibility. In some magnets, there are 2 spin degrees of freedom, but there is a preference for the spin to be aligned along (or, at an angle to) the crystal axes. Here is a Gibbs free energy with a 2-component field but lower symmetry:

$$G = \int d^dx \Big\{ \frac{1}{2} ((\vec{\nabla} m^x)^2 + (\vec{\nabla} m^y)^2) + \frac{1}{2} a_x (T - T_x)(m^x)^2 + \frac{1}{2} a_y (T - T_y)(m^y)^2 + \cdots \Big\} \tag{4}$$

If $T_x > T_y$, the first phase transition as will come down from high temperature will be at $T = T_x$. At this point, the m^x degree of freedom orders, while the m^y degree of freedom remains disordered. Another possibility is that the system has a symmetry between the two axes. If the system has the Z_4 or D_4 symmetry of the crystal point group – for our purposes, the symmetry of $90°$ rotations in the x-y plane



(5)

then the two A coefficients must be equal but there are now two possible quartic operators

$$\begin{aligned} G = \int d^dx \Big\{ & \frac{1}{2} ((\vec{\nabla} m^x)^2 + \vec{\nabla} m^x)^2) + \frac{1}{2} a(T - T_c)((m^x)^2 + (m^y)^2) \\ = \quad & + \frac{b}{4} ((m^x)^2 + (m^y)^2)^2 + \frac{c}{4} (m^x)^2 (m^y)^2 \Big\} \end{aligned} \tag{6}$$

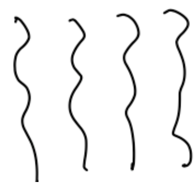
You can explore the physics of this system in this week's problem set.

A Heisenberg ferromagnet has $SO(3)$ symmetry. In the same way, one can write a Landau theory with $SO(n)$ symmetry,

$$G = \int d^dx \Big\{ \frac{1}{2} \sum_a (\vec{\nabla} m^a)^2 + \frac{1}{2} a(T - T_c) \sum_a (m^a)^2 + \frac{1}{4} b (\sum_a (m^a)^2)^2 \Big\} \tag{7}$$

where now the index a is summed over $1, \ldots, n$. Similarly to the previous case, we can break the symmetry by adding terms that are not invariant under the full $SO(3)$ but are invariant under the relevant crystal point groups, which are discrete subgroups of $SO(3)$. It is important to note that, for some point groups, there are no new invariants in the Landau theory. For example, in the XY model above, the symmetry group Z_6 (rotations by $\pi/3$) does not allow the quartic term proportional to $(m^x)^2(m^y)^2$, so the Landau theory is the same as that for the fully $SO(2)$ symmetric model. Again, this points to common features of those systesms that I will discuss in more detail later in the course.

In some systems, the order parameter has a vector character that can line up with spatial axes. An example is a system called a *nematic liquid crystal*. This is a liquid filled with long molecules



(8)

At a temperature T_c, there is an order-disorder transition that causes the molecules to line up macroscopically. The order parameter is a vector $\vec{n}$ giving the orientation of the molecules, with the directions

$$\vec{n} \text{ and } -\vec{n} \tag{9}$$

identified. Some terms in the Landau free energy are

$$\begin{aligned} G = \int d^3x \Big\{ & \frac{1}{2}\sum_a b(\vec{\nabla} n^a)^2 + \frac{\kappa}{2}(\vec{\nabla}\cdot\vec{n})^2 + a(T-T_c)(\vec{n})^2 \\ & + \frac{b}{4}(\vec{n}\cdot\vec{\nabla}\times\vec{n})^2 + \frac{c}{4}(\vec{n}\times\vec{\nabla}\times\vec{n})^2 \Big\} \end{aligned} \tag{10}$$

This formalism is used to describe some transitions among different orientation patterns in the textbook of Chaikin and Lubensky, *Principles of Condensed Matter Physics*, Chapter 6.

The Landau theories with continuous symmetries have a new feature that I will now discuss. Let's analyze this point by studying the $SO(3)$-symmetric theory (7) in more detail. For $T < T_c$, for the situation of a uniform magnetization, the minimization equation for m^a is

$$a(T-T_c)m^a + b(\sum_b (m^b)^2)m^a = 0 \tag{11}$$

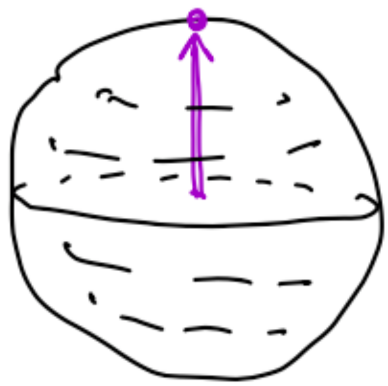
The nontrivial solutions satisfy

$$\sum_b (m^b)^2 = \frac{a(T_c - T)}{b} \ . \tag{12}$$

These solutions form a sphere in 3 dimensions. I will pick a particular point for close analysis

$$\vec{m} = (0, 0, m) \ , \tag{13}$$

but any overall $SO(3)$ rotation of this configuration will give an equivalent situation. This system then has a continuous space of degenerate ground states of the Gibbs free energy, which you can visualize as



(14)

All states on the sphere are equivalent. In 3 space dimensions, at least, each point corresponds to its own superselection sector.

Let's now compute the spin-spin correlation function in the model, still keeping $T < T_c$. To do this, expand about the uniform configuration

$$\vec{m}(x) = m_0 \ \hat{3} \qquad m_0 = \left[\frac{a(T_c - T)}{b}\right]^{1/2} \tag{15}$$

Let's write

$$\vec{m}(x) = (\pi^b(x), m_0 + \sigma(x)) \tag{16}$$

with $b = 1, 2$. The expansion of the Gibbs free energy is

$$\begin{aligned} G = \int d^x \Big\{ & \frac{1}{2}(\vec{\nabla}\pi^b)^2 + \frac{1}{2}(\vec{\nabla}\sigma)^2 \\ & + \frac{1}{2}a(T - T_c)m_0^2 + \frac{b}{4}m_0^4 + a(T - T_c)m_0\sigma + bm_0^3\sigma \\ & + \frac{a}{2}(T - T_c)\sigma^2 + \frac{a}{2}(T - T_c)(\pi^b)^2 + \frac{6b}{4}m_0^2\sigma^2 + \frac{2b}{4}m_0^2(\pi^b)^2 + \cdots \end{aligned} \tag{17}$$

where the sum over $b = 1, 2$ should be understood. The two terms in the second line linear in σ cancel by virtue of the value of m_0 given in (15). Remarkably, the quadratic terms in π^b in the third line also cancel,

$$\frac{a}{2}(T - T_c)(\pi^b)^2 + \frac{2b}{4}m_0^2(\pi^b)^2 = 0 \tag{18}$$

The terms with σ^2 do not cancel but rather lead to a positive quadratic term

$$\frac{a}{2}(T-T_c)\sigma^2 + \frac{6b}{4}m_0^2\sigma^2 = \frac{1}{2}(2a(T_c-T))\sigma^2 \ . \tag{19}$$

THis is the same expression that we found in the previous lecture, and so the σ fields have the same correlation length that we found there. However, for the π^b, the quadratic term is zero, and so we obtain a zero mass term and infinite range correlations for any low temperature.

Let's now use this information to compute the spin-spin correlation function. The complete correlation function

$$\left\langle m^a(x)m^b(y)\right\rangle \tag{20}$$

breaks up into four components with different behavior

$$\left\langle m^a(x)m^b(y)\right\rangle = \begin{pmatrix} \langle \pi^a(x)\pi^b(y)\rangle & \langle \pi^a(x)(m_0+\sigma(y))\rangle \\ \langle (m_0+\sigma(x))\pi^b(y)\rangle & \langle (m_0+\sigma(x))(m_0+\sigma(y))\rangle \end{pmatrix} \tag{21}$$

The off-diagonal terms are zero. The term in the lower right hand corner is

$$\left\langle m^3(x)m^3(y)\right\rangle = m_0^2 + \overline{G}(x,y) \ , \tag{22}$$

where $\overline{G}(x,y)$ is the Green's function in the low temperature phase studied in the previous lecture. The terms in the upper left-hand corner are

$$\left\langle \pi^a(x)\pi^b(y)\right\rangle = \delta^{ab} \ G_0(x,y) \ , \tag{23}$$

where $G_0(x,y)$ is the zero-mass Green's function, the solution of the equation

$$\beta \ (-\nabla_x^2) \ G_0(x,y) = \delta^{(d)}(x-y) \ . \tag{24}$$

Recall that this Green's function has only a power-law decay

$$G_0(x,y) = \frac{C}{|x-y|^{d-2}} \ , \tag{25}$$

even in the ordered state.

A way to understand this is to think of the minimization equation as a wave equation, adding a time varible to describe the dynamics of the π^b modes in time

$$\beta(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\pi^b(x) = 0 \tag{26}$$

This introduces a new constant c that depends on the physical system that is being modeled.

From this equation, we see that π is a wave with the dispersion relation

$$\omega(k) = ck \ . \tag{27}$$

When the system is quantized, the classical waves become quantum particles with energy

$$E(k) = \hbar ck \ . \tag{28}$$

These particles have *zero mass* (more generally, *zero energy gap*) and the speed c. They are called "Goldstone bosons" or "Nambu-Goldstone bosons" after Yoichiro Nambu and Jeffrey Goldsone, who studied this phenomenon in the late 1950's.

Goldstone bosons are an immediate consequence of the continuous multitude of ground states of the Gibbs free energy G. They are a classical phenomenon, but it is easier to write formulae if we go to a quantum-mechanical language. Let $|m_0\rangle$ be a particular ground state of G, and let $q(x)$ be an operator that generates the continuous symmetry on fields at the point x. The global rotation by the angle α is then implemented by the operator

$$U = \exp[i \int d^dx \ \alpha q(x)] \tag{29}$$

For example, for a magnet, $T(x)$ would be $\vec{S}_i(x)$, the angular momentum operator that rotates the spins. The operator U is not actually well-defined, since it connects different superselection sectors.

In the $SO(3)$ example, you will notice that the three symmetry generators split into two classes. The magnetized state $\vec{m} = (0, 0, m_0)$ is left unchanged by a rotation about the $\hat{3}$ axis, and so this rotation remains a symmetry of the ordered state. On the other hand, rotations about the $\hat{1}$ and $\hat{2}$ axes transform the ordered state into another possible minimum of G. These are the spontaneously broken symmetries.

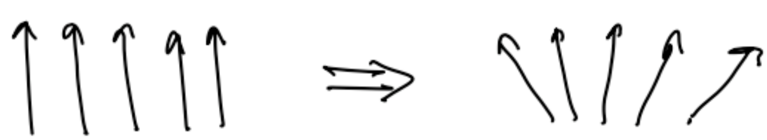
We can try to define U in (29) more sensibly by restricting the rotation to a large wavepacket $f(x)$,

$$f(x) = \cos(\vec{k} \cdot \vec{x}) e^{-x^2/2L^2} \tag{30}$$

Then, using the broken generator $S_1(x)$ to transform the state

$$U_{\vec{k}} = \exp[i \int d^dx \ \alpha f(x) S_1(x)] \tag{31}$$

we create a state in which the spin orientation changes slowly across the direction of $\vec{k}$,



(32)

Please notice that this spin rotation costs almost no free energy. As long as the magnitude of $\vec{m}(x)$ stays at the value m_0, the regions of rotated spins have exactly the same free energy as the original system, up to the energy cost of gradient in the spin direction from point to point. Thus if L is very large and we take $k \to 0$, we find a *spin wave* with an energy above the ground state that becomes as small as we wish as $k \to 0$,

$$\lim_{k\to 0} \omega(k) = 0 \ . \tag{33}$$

In the Landau theory we have

$$\omega(k) \sim k \tag{34}$$

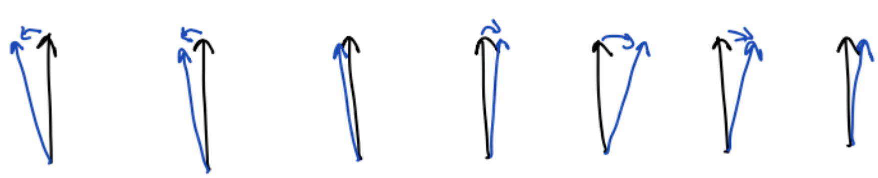
but other more complex system might have different power laws $\omega(k) \sim k^\alpha$. In any case, the modes of oscillation and the associated quantum particles have zero excitation energy in the limit $k \to 0$.

I have argued for *Goldstone's theorem*:

> When continuous symmetries of the Hamiltonian are spontaneously broken, for every broken symmetry there is a gapless mode of excitation (a massless particle or Goldstone boson).

These Goldstone bosons are often the most important quasiparticles in condensed matter systems at low temperature. We will see some specific examples of their effects next week.

We now ought to make a more sophisticate picture of a system with spontaneous breaking of a continuous symmetry. At temperatures well below T_c, the ordered state has a uniform magnetization modified by the presence of fluctuating Goldstone modes,



(35)

Let's try to represent the effect of these fluctuations mathematically. This is easiest to discuss in the $n = 2$ case with 1 Goldstone boson. Write the 2-dimensional local magnetization as a complex number

$$\vec{m}(x) = (m^1(x), m^2(x)) \quad \to \quad (m^1 + im^2)(x) \ . \tag{36}$$

so that

$$\langle \vec{m}(x) \cdot \vec{m}(y) \rangle = \left\langle m^1(x)m^1(y) + m^2(x)m^2(y) \right\rangle = \mathrm{Re}[m(x)m^*(y)] \ . \tag{37}$$

In the ordered phase, we can represent unperturbed magnetized state as a real value $m(x) = m_0$ and include the influence of Goldstone boson modes by adding a fluctuation imaginary part to $m(x)$,

$$m(x) = m_0 e^{i\pi(x)/m_0} \tag{38}$$

Then

$$\langle \vec{m}(x) \cdot] \vec{m}(y) \rangle = m_0^2 \mathrm{Re} \left\langle e^{i\pi(x)/m_0} e^{-i\pi(y)/m_0} \right\rangle \ . \tag{39}$$

We need to compute the expectation value of the random variables $\pi(x)$ in this expression using the probabilty distribution given by the Landau free energy

$$\int \mathcal{D}\pi \exp\Big[-\beta \int d^d x \frac{1}{2} (\vec{\nabla}\pi)^2 \Big] \tag{40}$$

This looks like an integral that is nontrivial to evaluate. However, it turns out that we can work this out using Wick's theorem.

As a model, let's evaluate $\langle e^x \rangle$, where x is a Gaussian random variable. I claim that the result is

$$\langle e^x \rangle = \exp[\frac{1}{2} \wick{\c{x} \c{x}}] \ . \tag{41}$$

Here is the proof: Expand

$$\langle e^x \rangle = \left\langle 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots \right\rangle \tag{42}$$

The odd terms are all zero, so

$$\langle e^x \rangle = \left\langle 1 + 0 + \frac{1}{2!} \langle x^2 \rangle + 0 + \frac{1}{4!} \langle x^4 \rangle + \cdots \right\rangle \tag{43}$$

A typical even term is

$$\frac{\langle x^{2n} \rangle}{n!} = \frac{(2n-1)(2n-3)\cdots(1)}{2n(2n-1)(2n-2)\cdots(1)} (\wick{\c{x} \c{x}})^n \ . \tag{44}$$

But

$$\frac{(2n-1)(2n-3)\cdots(1)}{2n(2n-1)(2n-2)\cdots(1)} = \frac{1}{2^n n!} \tag{45}$$

So indeed the expression resums to

$$\langle e^x \rangle = \sum_n \frac{(1/2)^n}{n!} (\wick{\c{x} \c{x}}]^n = \exp[\frac{1}{2} \wick{\c{x} \c{x}}] \ . \tag{46}$$

In the example at hand,

$$x = i(\pi(x) - \pi(y))/m_0 \tag{47}$$

so that

$$\left\langle e^{i(pi(x)-\pi(y))/m_0} \right\rangle =$$

$$\exp\left[-\frac{1}{2m_0^2}\{\langle\pi(x)\pi(x)\rangle + \langle\pi(y)\pi(y)\rangle - 2\langle\pi(x)\pi(y)\rangle\}\right]$$

$$\exp\left[-\frac{1}{m_0^2}\{G_0(0,0) - G_0(x,y)\}\right] \tag{48}$$

where I have used $G_0(x,x) = G_0(y,y) = G_0(0,0)$ by translation invariance. If you look at the explicit expressions, you will see that $G(x,0)$ diverges as $x \to 0$. However, this is an artifact of the continuum approximation that we have made. On a lattice, the divergence is cut off at the lattice spacing or atomic spacing a of the underlying physical system. For example, in 3 dimensions

$$G_0(x,0) = \frac{T}{4\pi|x|} \approx \frac{T}{4\pi a} \tag{49}$$

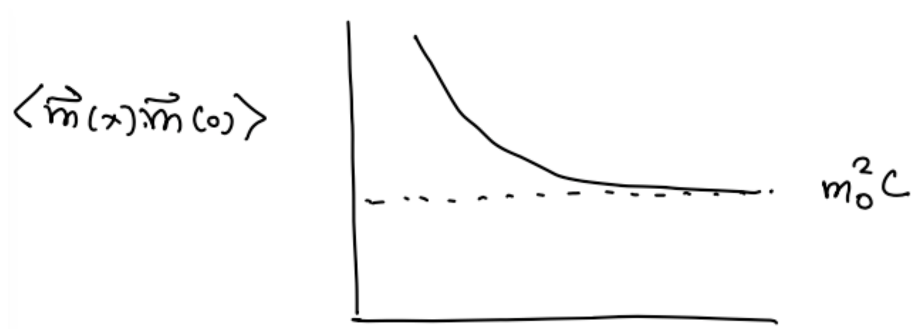
Let's account this as an overall constant in the expectation value, writing

$$\left\langle e^{i(\pi(x)-\pi(y))/m_0} \right\rangle = m_0^2 \cdot C \cdot \exp[G_0(x,y)/m_0^2] \ . \tag{50}$$

In 3 dimensions, this gives for the low-temperature behavior of the correlation function

$$\langle\vec{m}(x)\cdot\vec{m}(y)\rangle = m_0^2 \cdot C \cdot \exp\left[\frac{T}{4\pi m_0^2}\frac{1}{|x-y|}\right] \ . \tag{51}$$

The correlation function falls off as a result of the Goldstone fluctuations, but to a finite value,



(52)

Notice that the approach to the asymptote is not exponential but rather goes as $1/|x-y|$.

In 2 dimensions, we can carry out the same analysis, but we will find a very different result. In 2 dimensions, the Green's function is

$$G_0(x,y) = -\frac{T}{2\pi}\log|x-y| \ . \tag{53}$$

Then

$$\begin{aligned}\langle \vec{m}(x)\cdot\vec{m}(y)\rangle &= m_0^2\cdot C\cdot \exp\Big[-\frac{T}{2\pi m_0^2}\log|x-y|\Big]\\ &= m_0^2 C\ \frac{1}{|x-y|^{T/2\pi m_0^2}}\end{aligned} \tag{54}$$

As $|x-y|\to\infty$, the correlation function tends to zero! Thus, the Goldstone fluctuations destroy the magnetic order in the low-temperature phase of this model. Notice that the magnetic correlations become longer-ranged as the temperature is lowered, but at no finite temperature is there a correlation at infinite range.

In a previous lecture, I argued that there are no order-disorder transitions in 1 space dimension. We now see that the XY model has no order-disorder transition also in 2 dimensions. Any ordering at low temperature is wiped out by the fluctuations of the Goldstone modes.

What about other models? It can be proved that, for a spin model with n spin components, $n>2$,

$$\left\langle \vec{S}_I\cdot\vec{S}_J\right\rangle\Big|_n \le \left\langle \vec{S}_I\cdot\vec{S}_J\right\rangle\Big|_{n=2}\ . \tag{55}$$

Intuitively, if there are more fluctuating spin components, the correlation between spins should decrease. This inequality implies that there is also no magnetic order in 2 dimensions for any of the spin models with $n\ge 2$.

This result can be formalized and proved as a theorem, the *Mermin-Wagner Theorem*:

> In 2 dimensions, there cannot be an order-disorder transition that breaks a continuous symmetry of the Hamiltonian.

So, for the $n\ge 2$ models, the lower critical dimensionality is at $d=2$.