Dec. 13

Physics 212 – Statistical Mechanics

Conformal Invariance and Critical Exponents in 2 Dimension

In the previous lecture, I introduced the conformal group, and we studied its implications for scale-invariant models in 3 dimensions. In this lecture, I will discuss the implications of conformal invariance in 2 dimensions. We will see that the conformal group is actually larger in 2 dimensions, allowing even stronger and also more intuitive constraints.

I will work in 2-dimensional Euclidean space. It is convenient to parametrize point in this space by complex numbers

$$z = x + iy \qquad \overline{z} = x - iy \ . \tag{1}$$

The dot product of vectors is

$$\vec{v}_1 \cdot \vec{v}_2 = v_1^x v_2^x + v_1^y v_2^y \ . \tag{2}$$

If we set

$$v^z = v^x + iv^y \qquad v^{\overline{z}} = v^x - iv^y \ , \tag{3}$$

then

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{2}(v_1^z v_2^{\overline{z}} + v_1^{\overline{z}} v_2^z \ . \tag{4}$$

So this space has a slightly nontrivial metric $g_{\mu\nu}$, with

$$g_{z\overline{z}} = g_{\overline{z}z} = \frac{1}{2} \qquad g_{zz} = g_{\overline{z}\overline{z}} = 0 \ . \tag{5}$$

The energy-momentum tensor is symmetric, so it has three independent components

$$T_{zz} \ , \quad T_{\overline{z}\overline{z}} \ , \quad T_{z\overline{z}} \tag{6}$$

In the previous lecture, we wrote the equation for energy-momentum conservation as

$$\partial_\mu T^{\mu\nu} = 0 \tag{7}$$

In these coordinates

$$\partial_z T_{\overline{z}z} + \partial_{\overline{z}} T_{zz} = 0 \ , \qquad \partial_z T_{\overline{z}\overline{z}} + \partial_{\overline{z}} T_{z\overline{z}} = 0 \tag{8}$$

The condition for scale invariance is that $T_{\mu\nu}$ be traceless. In these coordinates, that condition reads

$$g^{\mu\nu}T_{\mu\nu} = 0 \qquad \text{or} \qquad T_{z\overline{z}} = 0 \tag{9}$$

The combination of (8) and (9) gives

$$\partial_{\overline{z}} T_{zz} = 0 \qquad \partial_z T_{\overline{z}\overline{z}} \tag{10}$$

That is, T_{zz} is a function only of z — an analytic function — and $T_{\overline{z}\overline{z}}$ is a function only of $\overline{z}$. From here on, I will write

$$T(z) = T_{zz} \qquad \overline{T}(\overline{z}) = T_{\overline{z}\overline{z}} \ . \tag{11}$$

This simplified condition for scale invariance admits a large number of additional symmetries. If we write

$$\epsilon^\lambda = (\epsilon^z, 0) \ , \tag{12}$$

then the requirement that $x^\mu \to x^\mu + \epsilon^\mu(x)$ is a symmetry

$$\partial^\mu (T_{\mu\nu}\epsilon^\nu) \tag{13}$$

becomes

$$\partial_{\overline{z}}(T_{zz}\epsilon^z) = 0 \qquad \text{or} \qquad \partial_{\overline{z}}\ \epsilon^z = 0 \tag{14}$$

Then ϵ^z can be a general function of z (but not $\overline{z}$), so that any analytic function of z generates a symmetry. The same argument goes through on the anti-analytic side; any transformation

$$\overline{z} \to \overline{z} + \overline{\epsilon}(\overline{z}) \ , \tag{15}$$

with $\overline{\epsilon}^{\overline{z}}$ a function only of $\overline{z}$, also generates a symmetry.

The result that scale invariant theories are invariant with respect to all analytic changes of variables should not be new to you. In 2-dimensional electrodynamics, the Laplace equation is

$$\nabla^2 \phi = 0 \qquad \text{or} \qquad \partial_z \partial_{\overline{z}}\ \phi = 0 \ , \tag{16}$$

so the solutions to the Laplace equation are

$$\phi = f(z) + g(\overline{z}) \ , \tag{17}$$

for any analytic function $f(z)$ and anti-analytic function $g(\overline{z})$. In undergraduate E&M, we require that ϕ is real-valued, so $g(\overline{z}) = (f(z))^*$. The contraction of scalar fields in 2 dimensions is

$$\langle \phi(x_1)\phi(x_2) \rangle = -\frac{1}{2\pi} \log |x_1 - x_2| \ . \tag{18}$$

To display the analytic structure, we can rewrite this as

$$\langle\phi(x_1)\phi(x_2)\rangle = -\frac{1}{4\pi}\Big\{\log(z_1 - z_2) + \log(\overline{z}_1 - \overline{z}_2)\Big\}. \tag{19}$$

For the results that I will explain in this lecture, I will work only with the analytic symmetries. In actual 2-dimensional statistical mechanics problems, the correlation functions are real-valued, and so one needs to knit together the analytic and anti-analytic functions. That brings in some extra subtleties that I will not have room to discuss here.

For convenience in this lecture, I would like to rescale the canonical scalar field ϕ so that the analytic part of its contraction is

$$\overbrace{\phi(z_1)\phi}(z_2) = -\log(z_1 - z_2)\ . \tag{20}$$

A spin-1/2 field obeying the Dirac equation has two components

$$\Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \tag{21}$$

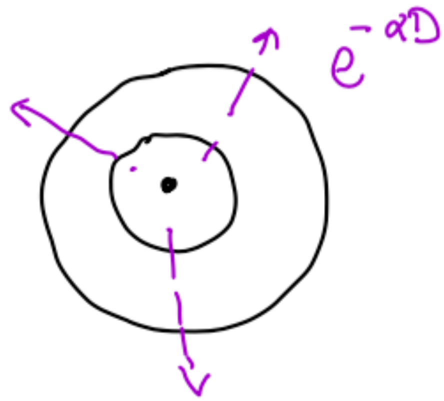
In the limit of zero mass, the Dirac equation becomes conformally-invariant. In the solution of the Dirac equation, the component ψ is an analytic field and the component $\overline{\psi}$ is an antianalytic field. The contractions are

$$\overbrace{\psi(z_1)\psi}(z_2) = \frac{1}{z_1 - z_2} \qquad \overbrace{\overline{\psi}(z_1)\overline{\psi}}(z_2) = \frac{1}{\overline{z}_1 - \overline{z}_2} \tag{22}$$

We see from these equations that the fields ϕ and ψ have dimensions 0 and 1/2 in 2 dimensions. Actually, ϕ is not single-valued, so it is more convenient to work with $\partial_z\phi$, which has dimension 1. We will see that this transforms as a primary conformal field. The energy-momentum tensors for these fields are

$$T = -\frac{1}{2}(\partial_z\phi)^2 \qquad T = -\frac{1}{2}\psi\partial_z\psi\ . \tag{23}$$

Using these two free fields as examples, I will now build of the Hilbert space structure of 2-dimensional conformal field theories. As I described in the previous lecture, we can build this Hilbert space using radial quantization, in which the spatial slices are spheres (here, circles) around the origin and the evolution from one slice to another is generated by the dilatation operator D.



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We can think of D as the Hamiltonian. The ground state of the system is the state created by the point at $x = 0$ with so operator positioned there. This state, $|0\rangle$, has

$$D\,|0\rangle = 0 \tag{25}$$

All other eigenvalues of D are positive, since this is needed so that all correlation functions $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ will fall off at large separation. A primary conformal field of dimension $D_{\mathcal{O}}$ placed at the origin will create a state

$$|\mathcal{O}\rangle = \mathcal{O}(0)\,|0\rangle \qquad \text{such that} \qquad D\;|\mathcal{O}\rangle = D_{\mathcal{O}}\;|\mathcal{O}\rangle\;\; . \tag{26}$$

In two dimensions, the conformal algebra splits into a product of two algebras, one generating analytic transformations and one generating anti-analytic transformations. So, in particular, the single D found in higher dimensions is replaced with

$$D = \mathbf{d} + \overline{\mathbf{d}} \tag{27}$$

where $\mathbf{d}$ is built from the energy-momentum tensor component T and $\overline{\mathbf{d}}$ is built from the energy-momentum tensor component $\overline{T}$. If d are the eigenvalues of $\mathbf{d}$ and $\overline{d}$ are the eigenvalues of $\overline{\mathbf{d}}$, then the full scaling dimension of an operator or state is

$$D = d + \overline{d}\;. \tag{28}$$

In the following, I will analyze only the operator $\mathbf{d}$ that acts on the analytic coordinate. There is a parallel structure generated by $\overline{\mathbf{d}}$.

The action of $\mathbf{d}$ on an analytic primary conformal field is

$$\mathcal{O}(z) \to e^{\alpha d}\,\mathcal{O}(e^{\alpha} z)\;. \tag{29}$$

The infinitesimal version of this transformation must be generated by the commutator of D with $\mathcal{O}(x)$, so we expect

$$[\mathbf{d}, \mathcal{O}(z)] = d\mathcal{O}(z) + z\cdot\partial\mathcal{O}(z)\;. \tag{30}$$

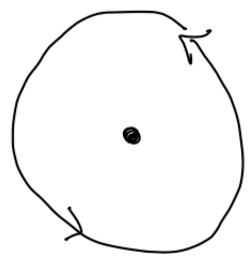
We can use this relation to learn how to build up $\mathbf{d}$ from $T(z)$. The generators of the analytic conformal symmetries should be integrals of $T(z)$ over the spatial slices. The operator

$$L_n = \oint \frac{dw}{2\pi i}\ T(w) w^{n+1} \tag{31}$$

should generate the transformation

$$z \to z + \alpha z^{n+1} \tag{32}$$

In particular, $\mathbf{d} = L_0$. The integral in (31) should be taken over the spatial slice, which is a circle around the origin,



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Notice that, because we are dealing with analytic functions, we are free to deform the contour to any other spatial slice. Thus, the L_n are conserved quantities. The infinite number of L_n operators generate the infinite-dimensional conformal symmetry corresponding to $\epsilon(z)$ in (12) being a general function of z.

The transformations (32) are generated by operators

$$\mathcal{L}_n = -z^{n+1}\partial_z \tag{34}$$

which have the commutation relations

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} \tag{35}$$

We will see, however, that these commutation relations do not capture the full complexity of a theory with an infinite number of fluctuating degrees of freedom.

We can actually see explicitly how the L_n generate the transformations (32) by examining the operator product expansion of $T(z)$ with the primary conformal fields of the free field theories described above.

Start with the scalar field theory. The OPE of T with the primary field $\partial_z\phi$. The singular terms in the OPE arise from contractions. In the limit $w \to z$

$$T(w)\ \ \partial_z\phi(z) = -\frac{1}{2}(\partial_w\phi)^2\ \ \partial_z\phi$$

$$
\begin{aligned}
&= -\frac{1}{2}\cdot 2\cdot \partial_w\phi\,\partial_w\overbrace{\phi\;\partial_z\phi} + \text{non-sing.} \\
&= -\partial_w\phi\partial_w\partial_z(-\log(w-z)) \\
&= \partial_w\phi\;\frac{1}{(w-z)^2}
\end{aligned}
\tag{36}
$$

We can move $\partial_w\phi(w)$ from w to z,

$$
\partial_w\phi(w) = \partial_z\phi(z) + (w-z)\partial_z^2\phi(z) + \cdots \tag{37}
$$

and this simplifies the operator product to

$$
T(w)\;\;\partial_z\phi = \frac{1}{(w-z)^2}\partial_z\phi + \frac{1}{(w-z)}\partial_z(\partial_z\phi) + \cdots \tag{38}
$$

The omitted terms have no singularity as $w \to z$.

In a similar way, we can work out the operator product expansion of T with $\psi(z)$

$$
\begin{aligned}
T(w)\;\;\psi(z) &= -\frac{1}{2}\psi\partial_w\psi\;\;\psi(z) \\
&= -\frac{1}{2}\psi(w)\partial_w\overbrace{\psi\;\psi}(z) - \frac{1}{2}\overbrace{\psi\partial_w\psi\;\psi}(z)
\end{aligned}
\tag{39}
$$

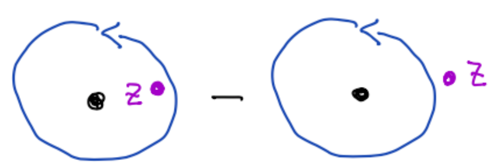
The Dirac fields are fermionic, and so they anticommute. This gives an extra minus sign in the second term. Then

$$
\begin{aligned}
T(w)\;\;\psi(z) &= \frac{1}{2}\psi(w)\frac{1}{(w-z)^2} + \frac{1}{2}\partial_w\psi\;\frac{1}{(w-z)} \\
&= \frac{1}{2}\frac{1}{(w-z)^2}\psi(z) + \frac{1}{2}\frac{1}{(w-z)}\partial_z\psi(z) + \frac{1}{2}\frac{1}{(w-z)}\partial_z\psi(z)
\end{aligned}
\tag{40}
$$

This simplifies to

$$
T(w)\;\;\psi = \frac{1/2}{(w-z)^2}\psi + \frac{1}{(w-z)}\partial_z(\psi) + \cdots \tag{41}
$$

We can now check that $\mathbf{d} = L_0$ using the results (38) and (41). The commutator of L_0 with $\mathcal{O}(z)$ is computed by placing these operators in different time orders,

$[L_0, \mathcal{O}(z)] =$ 

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The difference between these two expressions is given by the contour integral around the poles at $w = z$,

$$
\begin{aligned}
[L_0, \partial_z\phi] &= \oint \frac{dw}{2\pi i} \left\{ \frac{w}{(w-z)^2}\partial_z\phi + \frac{w}{(w-z)}\partial_z(\partial_z\phi) \right\} \\
&= 1 \cdot \partial_z\phi + z\partial_z(\partial_z\phi)
\end{aligned}
\tag{43}
$$

in agreement with (30) for this case with $d = 1$. In similar way,

$$
[L_0, \psi(z)] = \frac{1}{2} \cdot \psi + z\partial_z\psi \tag{44}
$$

which agrees with the dimension of the operator ψ, $d = 1/2$.

For a general conformal primary field $\mathcal{O}(z)$, the OPE of this operator with T must be

$$
T(w) \quad \psi = \frac{d}{(w-z)^2}\psi + \frac{1}{(w-z)}\partial_z(\psi) + \cdots \tag{45}
$$

to give the general commutation relation (30).

The complete (analytic-side) conformal group in two dimensions is generated by the complete set of conserved quantities L_n, with $n = -\infty, \cdots, \infty$. We can use the method of OPEs to work out the commutation relations of the L_n, which define the symmetry algebra. Let's compute this for the scalar field theory.

$$
\begin{aligned}
T(w) \quad T(z) &= (-\frac{1}{2}(\partial_w\phi)^2) \quad (-\frac{1}{2}(\partial_z\phi)^2) \\
&= \frac{1}{4}\Big[2 \cdot (\partial_w\overbracket{\phi \ \partial_z\phi})^2 + 4 \cdot \partial_w\phi\partial_w\overbracket{\phi \ \partial_z\phi}\partial_z\phi + \cdots \Big] \\
&= \frac{1/2}{(w-z)^4} + \partial_w\frac{(-1)}{(w-z)^2}\partial_z\phi \\
&= \frac{1/2}{(w-z)^4} + \frac{2}{(w-z)^2}(-\frac{1}{2}(\partial_w\phi)^2) + \frac{1}{(w-z)}\partial_z(-\frac{1}{2}(\partial_w\phi)^2) + \cdots
\end{aligned}
\tag{46}
$$

A similar calculation for the Dirac field—staying mindful of fermion exchange minus signs—gives

$$
\begin{aligned}
T(w) \quad T(z) &= (-\frac{1}{2}\psi\partial_w\psi) \quad (-\frac{1}{2}\psi\partial_z\psi) \\
&= \frac{1/4}{(w-z)^4} + \frac{2}{(w-z)^2}(-\frac{1}{2}\psi\partial_z\psi) + \frac{1}{(w-z)}\partial_z(-\frac{1}{2}\psi\partial_z\psi) + \cdots
\end{aligned}
\tag{47}
$$

The general pattern is

$$
T(w) \quad T(z) = \frac{c/2}{(w-z)^4} + \frac{2}{(w-z)^2}T(z) + \frac{1}{(w-z)}\partial_z T(z) + \cdots \tag{48}
$$

Notice that there is an extra term with respect to (45), propertional to the operator 1. The coefficient of this term, c, called the *central charge*, is a measure of the size of the theory, the total number of fluctuating degrees of freedom. For a theory with n scalar fields, for example, $c = n$. A massless fermion counts 1/2 of a massless scalar. The value of the central charge plays an important role in deeper properties of these conformal field theories. We will get a glimpse of that below.

To turn the OPE (48) into a commutation algebra, first compute the commutator of L_n with $T(z)$,

$$\begin{aligned}[L_n, T(z)] &= \int \frac{dw}{2\pi i}\, w^{n+1}\Big[\frac{c/2}{(w-z)^4} + \frac{2}{(w-z)^2}T(z) + \frac{1}{(w-z)}\partial_z T(z) + \cdots\Big] \\ &= \frac{(n+1)(n)(n-1)}{3!}\frac{c}{2}z^{n-2} + 2(n+1)z^n T(z) + z^{n+1}\partial_z T(z) \end{aligned} \tag{49}$$

Now integrate this with

$$\int \frac{dz}{2\pi i}\, z^{m+1} \tag{50}$$

and use

$$\int \frac{dz}{2\pi i}\, z^p = \delta(p, -1)\ . \tag{51}$$

We find

$$[L_n, L_m] = \frac{n(n^2-1)}{12}c\delta(n+m) + \int \frac{dz}{2\pi i}\, \Big\{2(n+1)z^{n+m+1}T(z) + z^{n+m+2}\partial_z T\Big\}\ . \tag{52}$$

We can simplify the last term by integrating by parts. Then, finally,

$$[L_n, L_m] = \frac{n(n^2-1)}{12}c\delta(n+m) + (n-m)L_{n+m} \tag{53}$$

This algebra agrees with (35) above, except that the earlier equation does not include the central charge term. The extended algebra (53) is called the *Virasoro algebra*, after Miguel Virasoro, who discovered this algebra in string theory.

It is noteworthy that the central charge term vanishes for $n = -1, 0, 1$. These generators correspond to translations, dilatations, and quadratic transformations, the operations that generate the conformal group in dimensions higher than 2.

Another property of the Virasoro generators L_n for $n \geq 0$ is that they annihilate the ground state $|0\rangle$. To calculate $L_n\,|0\rangle$, we can draw the contour around the origin. If the integrand of (31) is not singular at the origin, we can contract the contour and find

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By the same logic, the state $\langle 0|$, representing the point at infinity, the dual state to $|0\rangle$, is annihilated by L_{-n} for $n \geq -1$, since in these cases, we can contract the contour after an inversion that brings ∞ to the origin.

This observation is connected to a more general property. Notice that (53) implies

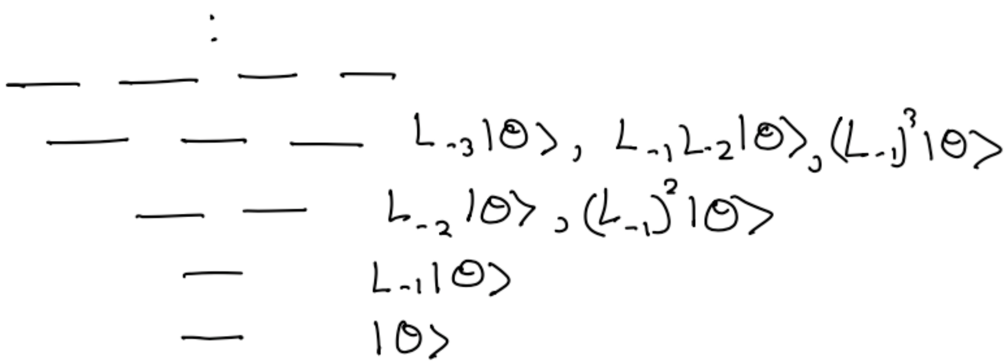
$$[L_0, L_n] = -nL_n \ . \tag{55}$$

This equation implies that, for $n > 0$, L_n lowers the eigenvalue of the dilatation generator $L_0 = \mathbf{d}$ by n units, and L_{-n} raises the eigenvalue of L_0 by n units. The L_0 eigenvalue must be positive (or 0 for $|0\rangle$), so this is another argument that $L_n |0\rangle = 0$. Similarly, the conformal primary field is the field of lowest dimension within its conformal representation. So, if $|\mathcal{O}\rangle = \mathcal{O}(0)\,|0\rangle$, then

$$L_n\,|\mathcal{O}\rangle = 0 \qquad \text{for all } n > 0 \tag{56}$$

Similarly, defining $\langle\mathcal{O}|$ by the state at ∞ after we push $\mathcal{O}(z)$ to large distances, $z \to \infty$, L_n must annihilate $\langle\mathcal{O}|$ for all $n > 0$. More generally, using the inversion as the adjoint operator as in higher dimensions, $(L_{-n})^\dagger = L_n$.

We can now see the structure of the representations of the conformal algebra in 2 dimensions. The states in the algebra are eigenstates of L_0. Lowest eigenvalue is given by the state $|\mathcal{O}\rangle$ associated with the primary conformal field. The higher states of the representation are given by applying the operators L_{-n} to $|\mathcal{O}\rangle$ in every possible combination,



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Since there are an infinite number of L_{-n} operators, we need only a small (sometimes even finite) number of primary fields to completely describe the degrees of freedom of a 2-dimensional conformal field theory.

Now we have enough information about conformal invariance in 2 dimensions to pose the same question that we posed in the previous lecture in 3-dimensional conformal field theories: Does the structure of the conformal group give us useful constraints on the spectrum of scaling dimensions? At a critical point in 2 dimensions, the system is invariant under the Virasoro algebra, and so the states of this system must form representations of this algebra. On the other hand, in a physical theory, the states must form a positive Hilbert space. In 1984, Alexander Polyakov, Alexander Belavin, and Alexander Zamolodchikov demonstrated that these ideas can combine to constraint the dimensions of primary fields. Not every choice of c and d leads to a positive-norm representation. They showed that we can use the positivity of the Virasoro representation to constraint the possible values of critical exponents.

To begin, consider the norm of the state $L_{-1}|O\rangle$, with d the dimension of $\mathcal{O}(z)$. This is

$$\| L_{-1}|\mathcal{O}\rangle \|^2 = \langle\mathcal{O}| L_1 L_{-1} |\mathcal{O}\rangle \tag{58}$$

Using $L_n|\mathcal{O}\rangle = 0$, this can be evaluated as

$$\| L_{-1}|\mathcal{O}\rangle \|^2 = \langle\mathcal{O}| [L_1, L_{-1}] |\mathcal{O}\rangle = \langle\mathcal{O}| \ 2L_0 \ |\mathcal{O}\rangle = 2d \ . \tag{59}$$

Positivity requires that this be positive, and so we have a direct proof that

$$d > 0 \ . \tag{60}$$

For $c \geq 1$, this is the strongest constraint that we find. Actually, for the case of a free scalar field at $c = 1$, we can create an operator of any positive dimension. Consider the operator

$$e^{i\alpha\phi(z)} \tag{61}$$

The correlation function of two such operators evaluates to

$$\left\langle e^{i\alpha\phi(z_1)} e^{i\alpha\phi(z_2)} \right\rangle = \exp[-\alpha^2 \log(z_1 - z_2)] = \frac{1}{(z_1 - z_2)^{\alpha^2}} \ . \tag{62}$$

From this, we see that the operator has dimension

$$d = \alpha^2/2 \ , \tag{63}$$

and so the dimensions of these operators can take any value $d > 0$.

However, if we choose $c < 1$, the story is more interesting. Let's examine the properties of the module (57) more explicitly. Begin at the second level, just above the state $L_{-1}|\mathcal{O}\rangle$. At this level, there are two states

$$L_{-2}|\mathcal{O}\rangle \ , \quad (L_{-1})^2 |\mathcal{O}\rangle \tag{64}$$

The norms of these states are

$$\| L_{-2}|\mathcal{O}\rangle \|^2 = \langle\mathcal{O}| L_2 L_{-2} |\mathcal{O}\rangle = \langle\mathcal{O}| 4L_0 + c/2 |\mathcal{O}\rangle = 4d + c/2 \tag{65}$$

and

$$\begin{aligned}\|L_{-1}L_{-1}\left|\mathcal{O}\right\rangle\|^2 &= \left\langle\mathcal{O}\right| L_1[L_1,L_{-1}]L_{-1}\left|\mathcal{O}\right\rangle + \left\langle\mathcal{O}\right| L_1L_{-1}[L_1,L_{-1}]\left|\mathcal{O}\right\rangle \\ &= \left\langle\mathcal{O}\right| L_1(2L_0)L_{-1}\left|\mathcal{O}\right\rangle + \left\langle\mathcal{O}\right| L_1L_{-1}(2L_0)\left|\mathcal{O}\right\rangle \\ &= 2d\cdot 2(d+1) + (2d)^2 = 8d^2+4d \end{aligned} \tag{66}$$

The overlap of the two states is

$$\left\langle\mathcal{O}\right| L_1^2L_{-2}\left|\mathcal{O}\right\rangle = \left\langle\mathcal{O}\right| L_1\ 3L_{-1}\left|\mathcal{O}\right\rangle = 2d\cdot 3 = 6d\ . \tag{67}$$

Then a linear combination of the two states

$$a\,L_{-2}\left|\mathcal{O}\right\rangle + b\,(L_{-1})^2\left|\mathcal{O}\right\rangle \tag{68}$$

has the norm equal to the matrix element

$$\begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} 4d+c/2 & 6d \\ 6d & 4d(2d+1) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{69}$$

If the determinant of this matrix is negative, then at least one such linear combination has negative norm. In that case, the representation cannot belong to a physical statistical mechanics model.

The determinant of the matrix has the value

$$32d\Big[d^2 - \frac{5-c}{8}d + \frac{c}{16}\Big] \tag{70}$$

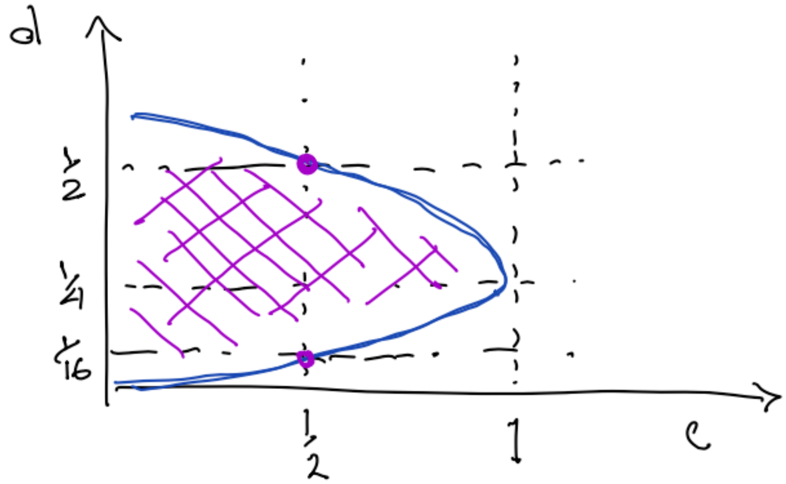
This cubic can easily be factorized. Before describing the general situation, let's write the quadratic form in brackets $c=1$,

$$d^2 - \frac{1}{2}d + \frac{1}{16} = (d-\frac{1}{4})^2\ . \tag{71}$$

Here the function is always ≥ 0. At $c = 1/2$, this function is

$$d^2 - \frac{9}{16} + \frac{1}{32} = (d-\frac{1}{16})(d-\frac{1}{2}) \tag{72}$$

For $1/16 < d < 1/2$ there must be a state in the module with negative norm, so these values of d cannot appear in a physical system For general c, the excluded region has the form



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We can continue this analysis to higher levels. Victor Kac found a general formula for the corresponding determinant at level N. Let p, q be integers such that $pq = N$. Then the zeros of the determinant all take the form

$$d_{pq} = d_0 + \left(\frac{p\alpha_+ + q\alpha_-}{2}\right) , \tag{74}$$

where

$$d_0 = -\frac{(1-c)}{24} \qquad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} . \tag{75}$$

For $c = 1$, $\sqrt{1-c} = 0$; then the pairs of zeros converge and the boundary curve is tangent to the line $c = 1$. Below $c = 1$, the region of d between the lower and upper zeros must have a state of negative norm. Eventually, as we consider higher and higher N, this argument seems to exclude all positive values of d in the region $c < 1$.

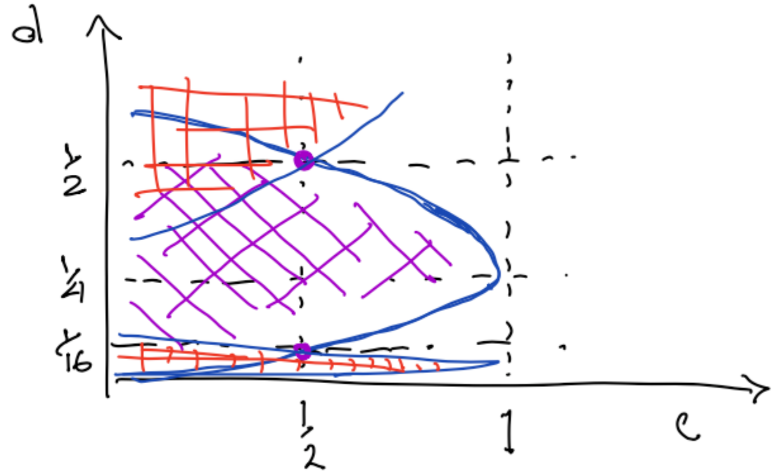
Howver, there is a way for a point in the (c, d) plane to evade this argument. If a state in the representation has zero norm at some level, the tower of states terminates and does not extend above that level. If the entire representation terminates, the constraints at higher levels have no force. The condition for this is that the boundary curves should cross at consecutive levels. This happens if

$$\frac{\alpha_-}{\alpha_+} = -\frac{m}{m+1} . \tag{76}$$

Then

$$((m+1)\alpha_+ + \alpha_-)^2 = (\alpha_+ + m\alpha_-)^2 , \tag{77}$$

so the representation has a zero at level $(m-1)$ and also at level m. We can see an example of this by comparing the boundary curves at levels 2 and 3,



$$\tag{78}$$

Daniel Friedan, Zongan Qiu, and Steve Shenker proved that the only physical models for $c < 1$ are the ones that satisfy this condition.

From this equation, we can solve for the allowed values of c. Since

$$\frac{\sqrt{25-c}-\sqrt{1-c}}{\sqrt{25-c}+\sqrt{1-c}} = \frac{m}{m+1} \tag{79}$$

we have

$$\sqrt{25-c} = (2m+1)\,\sqrt{1-c}\,, \tag{80}$$

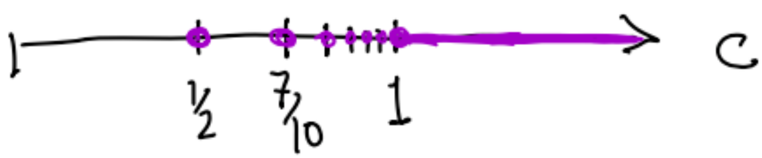
or

$$c = 1 - \frac{6}{m(m+1)}\,. \tag{81}$$

That is, the constrain that the representations of the Virasoro algebra are position for physical 2-dimension scale-invariant systems implies that, if $c < 1$, there can only be a discrete series of possible models. These have the c values

$$c = \frac{1}{2}\,,\ \ \frac{7}{10}\,,\ \ \frac{4}{5}\,,\ \cdots. \tag{82}$$

that is



$$\tag{83}$$

The allowed operator dimensions d for these models are also specified

$$\begin{aligned} c = \frac{1}{2} \ &: \qquad \frac{1}{16}, \frac{1}{2} \\ c = \frac{7}{10} \ &: \qquad \frac{1}{10}, \frac{3}{80}, \frac{3}{8}, \frac{7}{16}, \frac{3}{2} \end{aligned} \tag{84}$$

A parallel analysis applies for the anti-analytic algebra Operators in the full theory have analytic and anti-analytic scaling dimensions $(d, \overline{d})$ with the total scaling dimension being $D = d + \overline{d}$.

The value $c = 1/2$ corresponds to a model with one free massless fermion. I hope that you remember that the exact solution of the 2-dimensional Ising model showed that this system was exactly such a model. The allowed operator dimensions fit the pattern. We found that the operators of the Ising model are the free fermion operators, with dimensions $(d, \overline{d})$

$$\psi \ : \ (\frac{1}{2}, 0) \qquad \overline{\psi} \ : \ (0, \frac{1}{2}) \tag{85}$$

and the spin (order) and disorder operators, with

$$S \ : \ (\frac{1}{16}, \frac{1}{16}) \qquad \Sigma \ : \ (\frac{1}{16}, \frac{1}{16}) \tag{86}$$

These dimensions match the exact results quoted earlier. In particular, the total scaling dimension D of the spin operator is $1/8$, explaining the odd-looking result

$$\langle S(x) S(y) \rangle \sim \frac{1}{|x - y|^{1/4}} \tag{87}$$

that I quoted in my lecture on the 2-dimension Ising model.

The $c = 7/10$ model can be identified with a tricritical point in the Ising model. The $c = 4/5$ model can be identified with the 3-state Potts model, a model in which the spin can take one of three equivalent values. The remaining models in the series are identified with a set of model of surface fluctuations called the "Restricted Solid on Solid" (RSOS) models. In all cases, the values of the operator dimensions and the critical exponents are given exactly by the constraint that the representations of the Virasoro algebra that appear in the model should have positive norm.

We have now come a long way since the beginning of the course. We started with the basic ideas of Statistical Mechanics due to Boltzmann and Gibbs, 100 years old. We then found, successively, many new and profound ideas from the more recent history of physics:

- magnetic order and the critical point
- spontaneous symmetry breaking
- the analogy between quantum and statistical fluctuations
- the Landau description of spontaneously broken symmetry
- topologically stable field configurations
- Goldstone bosons
- superfluidity
- fermion pair condensation
- the Higgs mechanism
- scale-invariant dynamics
- integrating out and the renormalization group
- Feynman diagrams
- upper and lower critical dimensions
- conformal invariance

By now, there is an enormous superstructure of contemporary physics build on this foundation. I hope that this course will provide you with a point of entry into these new directions.