Physics 212 – Statistical Mechanics

Conformal Invariance and Critical Exponents

In the previous few lectures, we have been studying the behavior of thermodynamic functions in the vicinity of the critical point in the phase diagram of magnets and other system with order-disorder transitions. We found that these functions generally have non-analytic singularities at the critical point, with nontrivial power laws in $(T - T_c)$. I introduced the RG equations to compute these behaviors. We found that we could understand the critical point as a fixed point of the RG, giving rise to a scale-invariant statistical mechanical system. Close to the critical point, the fixed point is unstable in a direction corresponding to a small change in the temperature. We found that the rate of the instability gave the value of the critical exponent ν and that, more generally, the properties of operators in the scale-invariant theory gave us the full set of critical exponents.

In the previous lectures, we could compute the values of the critical exponents to the extent that we could write the exact RG equations. However, in the examples we studied, this seem to be possible only in special cases, for example, for continuous space dimensionality very close to 4 dimensions or 2 dimensions.

In this lecture and the next, I will discuss a different approach to the calculation of these exponents. I will argue that a scale-invariant statistical system is also invariant under a larger symmetry group called the conformal group. The requirement of invariance under conformal symmetry puts additional restrictions on the model, and these turn out to be very strong. In this lecture, I will explain how the use of conformal symmetry allows us to obtain very accurate values of the critical exponents in 3-dimensional magnetic models.

Let's review what we need to compute. A critical point of a magnetic model is described by a scale-invariant theory. For concreteness, I will discuss the scaling theory of the Ising model, with one spin degree of freedom. The operators of lowest dimension in this theory are, first, the spin operator $S(x)$, and second, the operator that multiplies the A or m^2 coefficient of Landau theory, $S^2(x)$. I will call this the "mass operator". Each of these operators has a *scaling dimension* D that describes its behavior under scale transformations

$$\mathcal{O}(x) \to \lambda^D \; \mathcal{O}(\lambda x) \tag{1}$$

such that the correlation function

$$\langle \mathcal{O}(x) \, \mathcal{O}(y) \rangle = \frac{A}{|x-y|^{2D}} \tag{2}$$

is invariant under scale transformations. In the simple case in which we take account of dimensional analysis rescaling only, the dimension of S is $(d-2)/2$ and the dimension of S^2 is $(d-2)$. However, we saw in the RG analysis that these dimensions can differ from their dimensional-analysis values due to the effects of interactions. I will refer to these dimensions of these operators as D_S and D_m. In general, D_S differs from its naive value and also $D_m \neq 2D_S$.

The connection to critical exponents is the following: The exponent η is determined by the equation

$$D_S = \frac{d-2+\eta}{2} \ . \tag{3}$$

In the statistical weight, the term $d^d x\, m^2 S^2$ must be scale invariant, so the combination $m^2 S^2$ scale with dimension d. Since S^2 has dimension D_m, the coefficient m^2 must scale as

$$m^2 \sim \lambda^{d-D_m} \tag{4}$$

Then the exponent ν is given by

$$\nu = \frac{1}{d-D_m} \tag{5}$$

The thermodynamic exponents α, β, γ, δ are determined from η and ν by hyperscaling, an assumption that holds between the upper and lower critical dimensions. So we can understand the values of all of the critical exponents if we can determine D_S and D_m. We must also verify that all other operators in the theory have $D > d$ and are therefore irrelevant in Wilson's sense.

Just to solidify our intuition, note that, with the scaling of dimensional analysis,

$$D_S = \frac{d-2}{2} \qquad D_m = 2D_S = d-2 \ , \tag{6}$$

giving

$$\eta = 0 \qquad \nu = \frac{1}{2} \ . \tag{7}$$

These are the Landau theory results. These values will be changed when we take account of interactions.

Now let's look more closely into the general predictions of scale invariance. I claim that, if a system is scale-invariant, it is also invariant under a larger set of symmetries, the conformal symmetries. I should warn you that these arguments are not airtight and that some subtle exceptions are known.[1] However, the conditions under which this analysis holds are very general.

[1]See A. Dymarsky et al., JHEP 02, 099 (2016), arXiv:1402.6322 [hep-th].

We are working in d-dimensional Euclidean space. However, as I explained earlier in the course, there is a sort of time evolution in statistical mechanics that is given by the transfer matrix. We can slice the space along one direction

(8)

and generate the full partition function by evolving from one slice to another with the transfer matrix

$$\mathrm{tr}\, e^{-\beta\mathcal{H}} = \mathrm{tr}\,(e^{-\epsilon\mathcal{H}})^N = \mathrm{tr} T^N \tag{9}$$

We discussed that this evolution is formally the same as time evolution in quantum mechanics, with t replaced by imaginary time

$$t = -i\beta \tag{10}$$

Thus, I can take over equations written for symmetries in ordinary time evolution to describe symmetries of a statistical mechanics problem.

With this understanding, space-time symmetries are generated by integrals of the energy-momentum tensor over a time slice. In this passage, I will write d for the *space* dimension; the dimension of the original statistical system, or the space-time dimension in this analysis, is $(d+1)$. Roman indices run over $1, \cdots, d$ and Greek indices run over the $(d+1)$ values $0, \cdots, d$.

Let $T^{\mu\nu}(x)$ be the energy-momentum tensor for this system. The total energy and momentum

$$E = \int d^d x\, T^{00} \qquad P^i = \int d^d x\, T^{0i} \tag{11}$$

are the generators of time and space translations

$$t \to t + a^0 \qquad x^i \to x^i + a^i \tag{12}$$

More generally, the charges

$$Q_\epsilon = \int d^d x\, T^{0\nu} \epsilon_\nu(x) \tag{13}$$

are the generators of general coordinate transformations

$$x^\mu \to x^\mu + \epsilon^\mu(x)\ . \tag{14}$$

These may or may not be symmetries of the theory, depending on whether these charges are conserved. In general, the charges will be conserved if

$$\partial_\mu(T^{\mu\nu}\epsilon_\nu(x)) = 0\ . \tag{15}$$

his condition implies

$$0 = \int d^d x \; \partial_\mu (T^{\mu\nu} \epsilon_\nu(x))$$
$$= \int d^d x \left\{ \frac{d}{dt}(T^{0\nu}\epsilon_\nu(x)) + \frac{d}{dx^i}(T^{i\nu}\epsilon_\nu(x)) \right\} \tag{16}$$

The second term here integrates to a surface term at infinity, which is zero with appropriate boundary conditions. Then

$$\frac{d}{dt} \int d^d x \; T^{0\nu}\epsilon_\nu = \frac{d}{dt} Q_\epsilon = 0 \; . \tag{17}$$

The energy momentum tensor $T^{\mu\nu}$ is a symmetric tensor. Its conservation is expressed by the equation

$$\partial_\mu T^{\mu\nu} = 0 \tag{18}$$

This implies that, for a constant vector a^μ,

$$\partial_\mu (T^{\mu\nu} a_\nu) = 0 \tag{19}$$

and this, in turn, by the argument above, implies the conservation of total energy and momentum. We might also note that, if $\omega_{\mu\nu}$ is a constant antisymmetric tensor, then

$$\partial_\mu (T^{\mu\nu}\, \omega_{\nu\lambda}\, x^\lambda)$$
$$= T^{\mu\nu}\, \omega_{\nu\lambda}\, \delta^\lambda_\mu = T^{\mu\nu}\, \omega_{\nu\mu} = 0 \; . \tag{20}$$

The final result is zero because $T^{\mu\nu}$ is symmetric and $\omega_{\mu\nu}$ is antisymmetric. This implies the conservation of

$$Q^{\mu\nu}_\omega = \int d^d x \left(T^{0\mu} x^\nu - T^{0\nu} x^\mu \right) \; , \tag{21}$$

In space-time, this operator is the generator of rotations and Lorentz transformations. In the original Euclidean space, Q_ω is the is the generator of $(d+1)$-dimensional rotations.

An infinitesimal scale transformation is the coordinate transformation

$$x^\mu \to x^\mu + \alpha x^\mu \; . \tag{22}$$

This is a symmetry if

$$\partial_\mu \Big(T^{\mu\nu} \; \alpha x_\nu \Big) = 0 \; . \tag{23}$$

That condition implies

$$0 = T^{\mu\nu} \alpha \delta_{\mu\nu} \tag{24}$$

or, more simply

$$\mathrm{tr}[T] = 0 \ , \tag{25}$$

that is, that the energy-momentum tensor should be traceless. This condition might be familiar to you from the statement that a relativistic gas such as a photon gas has a traceless energy-momentum tensor, corresponding to the condition

$$\mathcal{E} - 3p = 0 \ , \tag{26}$$

where $\mathcal{E}$ is the energy density and p is the pressure.

The condition for scale-invariance, that the energy-momentum tensor is traceless, implies some additional space-time symmetries. For b^μ a constant vector,

$$\begin{aligned} \partial_\mu \Big(T^{\mu\nu} [2x_\nu x\cdot b - x^2 b_\nu] \Big) \\ = T^{\mu\nu} \Big[2\delta_{\mu\nu} x\cdot b + 2x_\nu b_\mu - 2x_\mu b_\nu \Big] \ , \end{aligned} \tag{27}$$

The last two terms cancel due to the symmetry of $T^{\mu\nu}$, and the first term is zero if $T^{\mu\nu}$ is traceless. Then

$$\partial_\mu \Big(T^{\mu\nu} [2x_\nu x\cdot b - x^2 b_\nu] \Big) = 0 \ . \tag{28}$$

This implies that the set of $(d+1)$ transformations

$$x^\mu \to x^\mu + 2x^\mu x\cdot b - x^2 b^\mu \tag{29}$$

are also symmetries. These are called the *conformal transformations.* It can be shown that these are the general coordinate transformations that preserve the angles between vectors.

A finite conformal transformation can be written

$$x \to \frac{x^\mu - x^2 b^\mu}{1 - 2x\cdot b + x^2 b^2} \ . \tag{30}$$

Note that, for very large values $x \to \infty$, this transformation carries

$$x^\mu \to -\frac{b^\mu}{b^2} \tag{31}$$

and it carries the point $-b^\mu/b^2$ to infinity. Then the inversion of space-time

$$\mathcal{I}: \quad x^\mu \to \frac{x^\mu}{x^2} \tag{32}$$

is an element of the group of conformal transformations. Actually, we can recover all of the new symmetries from the operations of inversion and translation. If $T(-a)$ is the translation by a^μ, then the operation

$$\mathcal{I}\ T(-a)\ \mathcal{I} \tag{33}$$

gives

$$
\begin{aligned}
x^\mu &\to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} - a^\mu \\
&\to \frac{x^\mu/x^2 - a^\mu}{1/x^2 - 2x\cdot a/x^2 + a^2} = \frac{x^\mu - x^2 a^\mu}{1 - 2x\cdot a + a^2}
\end{aligned} \tag{34}
$$

which is the general conformal transformation. Conformal transformations thus generalize scale transformations by including the possibility of a spatial inversion.

The inclusion of spatial inversion in the allowed group of transformations may be familiar to you from your study of complex variable theory. In complex analysis, the point at infinity is treated as an ordinary point which, using what can called "conformal transformations" in that context, can be mapped into an ordinary point in the complex plane. The group of conformal transformations on the complex plane is larger than the group of conformal transformations in a general dimension $(d+1)$. I will discuss the extra consequences of that enlarged symmetry group in the next lecture.

The full set of symmetry generators of a scale-invariant theory is then

$$
\begin{aligned}
P_\mu &= \partial_\mu \qquad & M_{\mu\nu} &= x_\mu\partial_\nu - x_\nu\partial_\mu \\
D &= x^\mu\partial_\mu \qquad & K_\mu &= -2x_\mu x\cdot\partial + x^2\partial_\mu
\end{aligned} \tag{35}
$$

For the rest of this lecture, I will go back to Euclidean space and write d as the full Euclidean space dimensionality. Then Greek indices will run over $1,\cdots,d$. In (35), I have called the generator of scale transformations D, the "dilatation" operator.

The algebra of the transformations (35) in d-dimensional Euclidean space is

$$
\begin{aligned}
[M_{\mu\nu}, P_\lambda] &= \delta_{\nu\lambda}P_\mu - \delta_{\mu\lambda}P_\nu \qquad [M_{\mu\nu}, K_\lambda] = \delta_{\nu\lambda}K_\mu - \delta_{\mu\lambda}K_\nu \\
[M_{\mu\nu}, M_{\lambda\sigma}] &= \delta_{\nu\lambda}M_{\mu\sigma} - \delta_{\mu\lambda}M_{\nu\sigma} - \delta_{\mu\sigma}M_{\mu\lambda} + \delta_{\mu\sigma}M_{\nu\lambda} \\
[D, P_\mu] &= P_\mu \qquad\qquad [D, K_\mu] = -K_\mu \\
[K_\mu, P_\nu] &= 2\delta_{\mu\nu}D - 2M_{\mu\nu}
\end{aligned} \tag{36}
$$

Formally, this extends the $SO(d)$ algebra of Euclidean space rotations to the group $SO(d+1,1)$.

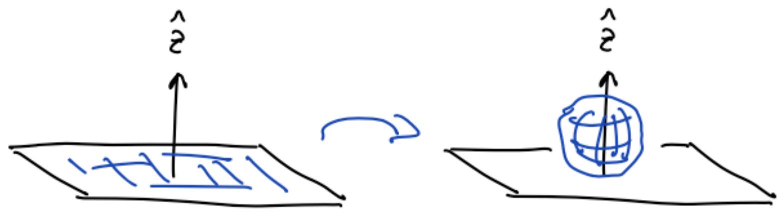
In 3 dimensions, the conformal transformation with $b^\mu = (0,0,-1/2)$,

$$
\vec{x} \to \frac{\vec{x} + \hat{z}x^2/2}{1 + \hat{z}\cdot x + x^2} \tag{37}
$$

carries the plane $x^\mu = (x, y, 0)$, with $r^2 = (x^2+y^2)$, into

$$
\frac{(x, y, r^2/2)}{1 + r^2/4} \ . \tag{38}
$$

which is a sphere of radius 1. The point $(x, y, z) = (0, 0, 0)$ is mapped back to the origin, and the point at $(x, y) \to \infty$ is mapped to (0,0,2),



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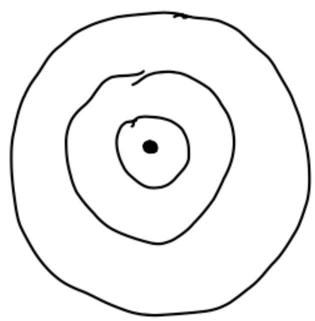
In general, conformal transformations carry planes into spheres. So we can generalize our notion of slicing the space with flat surfaces of definite time to slicing the space with spheres of definite radius.



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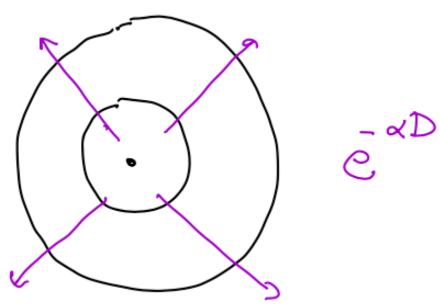
The transformation that takes one sphere into another is a dilatation. It is possible to set up a Hilbert space using this point of view. The analog of Minkowski time translation $e^{-it\mathcal{H}}$ is $e^{-\alpha D}$



(41)

and the operator D plays the role of the Hamiltonian. Notice from the commutation relations above that P^μ increases the value of D. This is natural, because

$$[P_\mu, \mathcal{O}(0)] = \partial_\mu \mathcal{O}(0) \ , \tag{42}$$

mapping $\mathcal{O}$, with dimension D into an operator of higher dimension $(D+1)$. Conversely, K^μ decreases the value of D. The initial state is the functional integral of fields around $x^\mu = 0$; we can call this $|0\rangle$. This state is dilatation-invariant,

$$D\,|0\rangle = 0 \tag{43}$$

If we put an operator at $x^\mu = 0$,

$$\mathcal{O}(0)\ \ |0\rangle = |\mathcal{O}\rangle\ \ , \tag{44}$$

then

$$D\ \ |\mathcal{O}\rangle = D_{\mathcal{O}}\ \ |\mathcal{O}\rangle\ \ , \tag{45}$$

where D is the dimension of the operator. This setup is called *radial quantization*. A full, coherent introduction to this formalism can be found in the TASI lectures of David Simmons-Duffin, arXiv:1602.07982.

Since the dimension D appears in the correlation function

$$\langle \mathcal{O}(x)\ \mathcal{O}(y)\rangle = \frac{A}{|x-y|^{2D}}\ , \tag{46}$$

the dimension D must be positive. Thus, there is a lowest value of D in every representation of the conformal algebra. This value belongs to the *primary state*, which is created by a *primary field*,

$$|D\rangle = \mathcal{O}_D(0)\ |0\rangle \tag{47}$$

The primary state must statisfy

$$K_\mu\ \ |D\rangle = 0 \quad \text{or} \quad (-2x_\mu x\cdot\partial + x^2\partial_\mu)\mathcal{O}\Big|_{x=0} = 0\ . \tag{48}$$

Then we can think of the operators P^μ and K^μ as raising and lowering operators for the Hamiltonian D. Operators created by acting P^μ, or translations, on the primary field are called *descendants*. These are

$$\partial_\mu \mathcal{O}(x)\ ,\quad \partial_\mu\partial_\nu\mathcal{O}(x)\ ,\cdots\ . \tag{49}$$

Given a primary field $\mathcal{O}_{\mathcal{D}}(x)$, we can build up the correlation functions of its descendants using the conformal group. This puts strong constraints on the form of correlation functions, as we will see in a moment.

Simply counting the dimensions, a two-point correlation function of primary fields with dimensions D_1 and D_2 must have the form

$$\langle \mathcal{O}_1(x)\ \mathcal{O}_2(y)\rangle = \frac{A}{|x-y|^{D_1+D_2}} \tag{50}$$

However, we can use a conformal transformation to move x close to infinity and then a translation to move y to zero. Then, near 0, we have the state

$$\mathcal{O}_2(0)\,|0\rangle = |D_2\rangle \qquad D\ \ |D_2\rangle = D_2\ \ |D_2\rangle \tag{51}$$

At infinity we have the state

$$\langle 0|\ \ \mathcal{O}_1(x) = \langle D_1| \qquad \langle D_1|\ \ D = \langle D_1|\ \ D_1 \tag{52}$$

These states can overlap only if then have same D eigenvalue. Thus, for the correlation function to be nonzero, we must have $D_1 = D_2$. The nonzero 2-point correlation functions are therefore all of the form

$$\langle \mathcal{O}_i(x)\ \mathcal{O}_j(y)\rangle = \frac{A_{ij}}{|x-y|^{2D_i}}\ , \tag{53}$$

where Aij has nonzero matrix elements only betwen operators with the same dimension D_i. It is very convenient to take linear combinations of the $\mathcal{O}_j$ and rescale so that

$$\langle \mathcal{O}_i(x)\ \mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2D_i}}\ . \tag{54}$$

Having now put the 2-point correlation functions of primary fields into a very simple form, we can ask if we can similarly simplify higher-point correlation functions. For the 3-point correlation function of primary fields, this can be done. Consider

$$\langle \mathcal{O}_1(x_1)\ \mathcal{O}_2(x_2)\ \mathcal{O}_3(x_3)\rangle\ . \tag{55}$$

Using a conformal transformation, we can send $x_3 \to \infty$. Then, with a translation, we can set $x_1 \to 0$. Finally, with a rotation and a rescaling, we can set $x_2 \to 1$. Let the value of the correlation function at this point be

$$\langle \mathcal{O}_1(0)\ \mathcal{O}_2(1)\ \mathcal{O}_3(\infty)\rangle = f_{123} \tag{56}$$

Then, undoing the transformations, we find a unique value for (55),

$$\langle \mathcal{O}_1(x)\ \mathcal{O}_2(y)\ \mathcal{O}_3(z)\rangle = \frac{f_{123}}{|x_1-x_2|^{D_1+D_2-D_3}|x_2-x_3|^{D_2+D_3-D_1}|x_3-x_1|^{D_3+D_1-D_2}}\ . \tag{57}$$

Further, since we can move any of the three points to any of the three positions $0, 1, \infty$, the tensor f_{123} must be totally symmetric in the three arguments.

To understand better the structure of (57), consider the limit in which $x_3 \to x_2$. The the operators $\mathcal{O}_2$ and $\mathcal{O}_3$ merge into a single effective operator located at x_3. According to (54), in order for this combined operator to overlap with $\mathcal{O}_1$, it must have a term proportional to the primary operator $\mathcal{O}_1$. Taking this limit in (57), we find

$$\langle \mathcal{O}_1(x_1)\ \mathcal{O}_2(x_2)\ \mathcal{O}_3(x_3)\rangle = \frac{1}{|x_1 - x_3|^{2D_1}} \left(\frac{f_{123}}{|x_2 - x_3|^{D_2+D_3-D_1}} \right) . \tag{58}$$

So (57) has just the form needed to demonstrate this required structure. The power law with which $\mathcal{O}_2$ and $\mathcal{O}_3$ combine is just that expected from dimensional analysis using the exact scaling dimensions.

More generally, two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(y)$, as their locations merge in the limit $y \to x$, can be written as a single operator located at x to compute correlation functions with other operators placed at a distance from x that is large compared to —x-y—. Writing this as an equation,

$$\lim_{x\to y}\ \mathcal{O}_1(x)\ \mathcal{O}_2(y) = \sum_k\ C_{12k}(x-y)\mathcal{O}_k(y) \tag{59}$$

The sum can run over all operators in the theory. This statement is called the *Operator Product Expansion* or OPE. The idea of the OPE is similar to the idea of expressing an arbitrary charge distribution, viewed from far away, as a sum of multipoles.

The operator product expansion was originally introduced by Kenneth Wilson. This concept fits together naturally with the integrating out operation and the renormalization group. Essentially, when two operators come together to the cutoff scale, integrating out the separate points gives an effective operator that is the composite of the two original operators.

If $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_k$ are primary conformal fields, the form of the coefficient function $C_{12k}(x-y)$ is fixed by the scaling dimensions of the three operators to be a well-defined power law. The only freedom is the overall coefficient f_{12k}. Then the OPE of primary conformal fields reads

$$\lim_{x\to y}\ \mathcal{O}_1(x)\ \mathcal{O}_2(y) = \sum_k\ \frac{f_{12k}}{|x-y|^{D_1+D_2-D_k}}\mathcal{O}_k(y) + \cdots\ , \tag{60}$$

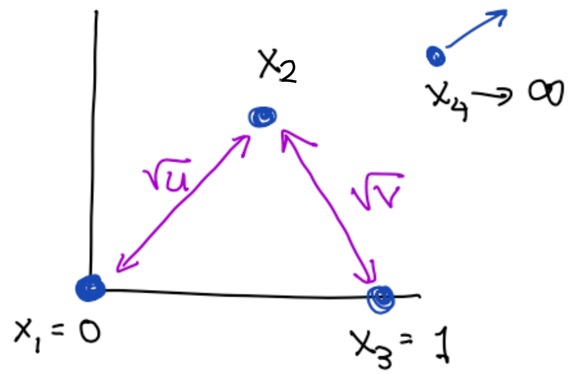
The omitted terms involve the descendants of $\mathcal{O}_k$ can can be worked out using the conformal algebra once the OPE coefficients f_{ijk} are given. By the logic above, f_{ijk} is a totally symmetric function of its three arguments.

Let's move now to the 4-point correlation function of primary fields. The simplicity of the 2- and 3-point functions does not extend to this case. Instead, there is a good deal more structure, but also a new and interesting set of constraints.

We wish to evaluate the 4-point correlation function of primary fields,

$$\langle \mathcal{O}_1(x_1)\ \mathcal{O}_2(x_2)\ \mathcal{O}_3(x_3)\ \mathcal{O}(x_4)\rangle\ . \tag{61}$$

Let's use a conformal transformation to send x_4 to infinity, then use a translation to send x_1 to 0, then use a rotation and rescaling to send x_3 to 1, and, finally, use a rotation about the x_1-x_3 axis to bring x_2 into a fixed plane,



(62)

There are still two degrees of freedom that have not been fixed, the distances between x_1 and x_2 and between x_2 and x_3. Call the squares of the distances u and v. We can define u and v more invariantly by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{63}$$

where $x_{12} = (x_1 - x_2)$, etc. These quantities are called *conformal cross-ratios*. It is not difficult to show that they are invariant to conformal transformations. Setting $x_1 = 0$, $x_3 = 1$, $x_4 \to \infty$, we see that these expressions equal the squares of the distances in the figure (62). The general form of the 4-point function can then be written

$$\langle \mathcal{O}_1 \; \mathcal{O}_2 \; \mathcal{O}_3 \; \mathcal{O}_4 \rangle = \frac{1}{|x_{12}|^{D_1+D_2} |x_{34}|^{D_3+D_4}} \left| \frac{x_{24}}{x_{14}} \right|^{D_1-D_2} \left| \frac{x_{14}}{x_{13}} \right|^{D_3-D_4} g(u,v) \; , \tag{64}$$

where $g(u,v)$ is a function of u and v that is still to be determined.

This is a large amount of structure. In 1970, Alexander Polykov conjectured that this structure was sufficient to completely constrain the values of the critical exponents. This idea is called the "conformal boostrap". However, it took a long time for that idea to be realized. A new idea was injected into this program in 2008 by Riccardo Rattazzi, Vyacheslav Rychkov, Erik Tonni, and Alessadro Vichi, and this idea was refined by a number of additional authors, in particular, Sheer El-Showk, Miguel Paulos, David Poland, and David Simmons-Duffin.

The idea begins with facts known in 1970. We can formally evaluate the 4-point function by taking the operator product expansion of $\mathcal{O}_1$ and $\mathcal{O}_2$, taking the operator product expansion of $\mathcal{O}_3$ and $\mathcal{O}_4$, and then evaluating the 2-point function of the two fused operators. If we look only at the contribution of one scalar primary operator $\mathcal{O}_k$ to the right-hand side of each operator product, these must be the same operator, and we find the contribution

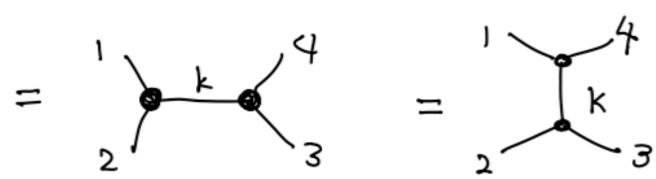
$$\langle \mathcal{O}_1 \; \mathcal{O}_2 \; \mathcal{O}_3 \; \mathcal{O}_4 \rangle \Big|_k = \left(\frac{f_{12k}}{|x_{12}|^{D_1+D_2}} \right) \left(\frac{f_{34k}}{|x_{34}|^{D_3+D_4}} \right) \left| \frac{x_{12} x_{34}}{x_{24}^2} \right|^{D_k} . \tag{65}$$

If we add the contributions from the descendants of the operator $\mathcal{O}_k$, we obtain an expression of the form

$$\langle \mathcal{O}_1 \ \mathcal{O}_2 \ \mathcal{O}_3 \ \mathcal{O}_4 \rangle \Big|_k = \frac{f_{12k} f_{34k}}{|x_{12}|^{D_1+D_2} |x_{34}|^{D_3+D_4}} \left| \frac{x_{24}}{x_{14}} \right|^{D_1-D_2} \left| \frac{x_{14}}{x_{13}} \right|^{D_3-D_4} g_k(u,v) \ . \tag{66}$$

The function $g_k(u,v)$ is not simple, but it can be determined explicitly from the properties of the conformal group. It depends only on dimensions of the operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ and $\mathcal{O}_k$ and, in the most general case, the spins of these operators. These functions, called *conformal blocks*, are known explicitly from analyses in the 1990's by Francis Dolan and Hugh Osborn. But at this stage we are still lacking the basic knowledge of the spectrum of operators, their scaling dimensions D_i, and their OPE coefficients f_{ijk}.

However, the 4-point function can be evaluated by this method in a different way, by contracting $\mathcal{O}_1$ with $\mathcal{O}_4$ and $\mathcal{O}_2$ with $\mathcal{O}_3$. These two evaluations must give the same answer.

$\langle S_1 S_2 S_3 S_4 \rangle =$  (67)

This constraint, called *crossing symmetry*, is the key to the conformal boostrap.

Let's now specialize to the case of the 4-point correlation function of spin operators

$$\langle S(x_1) \ S(x_2) \ S(x_3) \ S(x_4) \rangle \ . \tag{68}$$

The first evaluation in terms of conformal blocks, using the 12 and 34 OPEs, gives

$$\sum_k \frac{f_{SSk}^2}{|x_{12}^2 x_{34}^2|^{D_S}} \ g_k(u,v) \tag{69}$$

The second evaluation in terms of conformal blocks, using the 14 and 23 OPEs, gives

$$\sum_k \frac{f_{SSk}^2}{|x_{14}^2 x_{23}^2|^{D_S}} \ g_k(v,u) \ , \tag{70}$$

noting the fact that interchange of 1 and 3 interchanges u and v. The crossing relation follows from equating these expressions. Multiplying also by $|x_{13}^2 x_{24}^2|^{D_S}$, we find

$$\sum_k f_{SSk}^2 \left| \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \right|^{D_S} g_k(u,v) = \sum_k f_{SSk}^2 \left| \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2} \right|^{D_S} g_k(v,u) \ , \tag{71}$$

or, more simply,

$$\sum_k f_{SSk}^2 \; u^{-D_S} \; g_k(u,v) = \sum_k f_{SSk}^2 v^{-D_S} \; g_k(v,u) \; . \tag{72}$$

Now let's consider the various possibilities for $\mathcal{O}_k$. First of all, we can interpret the nonzero 2-point function

$$\langle S(x_1)\; S(x_2)\rangle = \frac{1}{|x_{12}|^{2D_S}} \tag{73}$$

as the statement that the OPE of two $S(x)$ operators contains the operator $\mathbf{1}$,

$$S(x_1)\; S(x_2) \sim \frac{1}{|x_{12}|^{2D_S}} \; \mathbf{1} \tag{74}$$

The operator $\mathbf{1}$ has $D = 0$. This operator has no dependence on x and so no descendants. We can isolate this term in each sum and rewrite the crossing relation as

$$u^{-D_S} + \sum_{k\neq 1} f_{SSk}^2 \; u^{-D_S} \; g_k(u,v) = v^{-D_S} + \sum_{k\neq 1} f_{SSk}^2 v^{-D_S} \; g_k(v,u) \; . \tag{75}$$

or

$$1 = \sum_{k\neq 1} f_{SSk}^2 \frac{v^{-D_S} g_k(v,u) - u^{-D_S} g_k(u,v)}{u^{-D_S} - v^{-D_S}} \; . \tag{76}$$

The mass operator $\mathcal{O}_m = S^2(x)$ also appears in the OPE, and, by our original assumptions, this is the nontrivial operator $\mathcal{O}_k$ of lowest dimension. It can be shown that higher-dimension operators appearing in the OPE give diminishing contributions to the right-hand side. On the other hand the right hand side is potentially a function of the variables u and v while the left-hand side is constant. I remind you that the function $g_k(u,v)$ is known for given D_S and D_k. A substantial conspiracy is necessary to cancel all of the u and v-dependence, and it may not be possible to arrange this. A particular complicating feature is that each term on the right-hand side appears with a coefficient f_{SSk}^2 that is unknown at this stage but is necessarily positive.

It is possible to make the constraint more explicit in the following way: Imagine fixing D_S, D_m, and a large set of increasing values D_k. For each term k, evaluate the conformal block at a number of different values of u and v. Imagine these values as a large vector $\vec{V}_k$. The most prominent of the vectors will be $\vec{V}_m$ associated with the operator $\mathcal{O}_m$. The crossing relation then reads

$$\vec{1} = \sum_{k\neq 1} \; f_{SSk}^2 \; \vec{V}_k \tag{77}$$

where $\vec{1}$ is the vector all of whose components equal 1 and the $\vec{V}_k$ are vectors that decrease with the assumed size of the dimension D_k. It may be possible to show that, for fixed D_S and D_m, additional vectors added with positive coefficients cannot be arranged to cancel all of the variations in the components of the vector $\vec{V}_m$ to allow the right-hand side of (77) to agree with the left-hand side. Then the chosen values (D_S, D_m) cannot appear in a consistent conformal field theory. The search for an optimal sum of fixed vectors is a subject called *linear programming* for which an extensive set of numerical methods has been developed to solve optimization problems for very large systems. Applying these methods, the authors listed above were able to eliminate large swaths of the (D_S, D_m) space. Very recently (2016), they were able to restrict the dimensions (D_s, D_m) of the S and S^2 operators in the scaling theory of the 3-dimensional Ising model to tiny regions of the (D_S, D_m).

Here are the current estimates: For the Ising $(n = 1)$ case in 3 dimensions,

$$D_S = 0.5181489(10) \qquad D_m = 1.412625(10)\ . \tag{78}$$

These values correspond to

$$\eta = 0.036298 \qquad \nu = 0.62997\ . \tag{79}$$

The same method can be applied to scaling theories of spin models with more components. For the XY model, the $n = 2$ case with $U(1)$ symmetry, in 3 dimensions,

$$D_S = 0.51926\,(32) \qquad D_m = 1.5117\,(25)\ . \tag{80}$$

These values correspond to

$$\eta = 0.0385 \qquad \nu = 0.6753\ . \tag{81}$$

For the Heisenberg model, the $n = 3$ case with $SO(3)$ symmetry, in 3 dimensions,

$$D_S = 0.51928\,(62) \qquad D_m = 1.5957\,(56) \tag{82}$$

These values correspond to

$$\eta = 0.0386 \qquad \nu = 0.7121\ . \tag{83}$$

These values are in excellent agreement with the determinations of 3-dimensional critical exponents quoted earlier in the course. The bootstrap analysis constitutes an amazing solution to this very difficult problem.