Physics 212 – Quiz #2

(issued Wednesday, November 16; due Wednesday, November 23)

I will use the quizzes and the final in this course to assign grades, keeping in mind that anyone who hands in all of the problem sets is assured a reasonable grade in the course. Because these quizzes will be graded, please abide by these rules:

- The quizzes are open-book. You may use any reference resources that you find. You may use your mathematical software such as MatLab or Mathematica. However, please do not collaborate with other students or ask help from other people—except that, if you have any questions about the quiz, please feel free to email me (mpeskin@slac.stanford.edu).
- The quizzes are posted at the course web site:

  `https://www.slac.stanford.edu/~mpeskin/Physics212/`

  This is the cover page. Please hand in your solution (upload to Gradescope) within 24 hours of the time that you turn the page and begin to solve the quiz.
- Please write on your solution: "I acknowledge the Stanford Honor Code." and sign it.
- The quizzes should be turned by the announced due date. If this is a problem for you, please email me.

There will be 2 quizzes in all. Each quiz will be worth 25 points. Partial credit will be given. The final will be worth 50 points.

1. The Fermi liquid He^3 has superfluid phases at milli-Kelvin temperatures, associated with Cooper pairing of He^3 atoms. This system is more complex than that of electrons in a metal in several ways. The most important is that, because He^3 atoms have hard-core repulsion in addition to Pauli exclusion, the form of the Cooper pair wavefunction is changed. In a superconductor, the quantity that obtains an expectation value in a uniform medium is

$$\Phi_{ab}(x) = \int \frac{d^3k}{(2\pi)^3}\; a_{\vec{k},a} a_{-\vec{k},b} \tag{1}$$

with a spherical integral ove the region around the Fermi surface. In this sense, the Cooper pair has an S-wave wavefunction. In He^3, the field that obtains an expectation value is

$$\Phi_{m,ab}(x) = \int \frac{d^3k}{(2\pi)^3}\; Y_{1m}(\hat{k})\; a_{\vec{k},a} a_{-\vec{k},b} \tag{2}$$

with a P-wave pair wavefunction that is zero for zero separation in $\vec{x}$ space.

In this quiz, we will work with a simplified model of He^3. In the real system, there is a coupling between spin and orbital angular momenta that is crucial to some properties of the superfluid phases. Here, we will ignore that coupling.

(a) Show that, in the case of an electron pair condensate, Φ_{ab} vanishes if (ab) forms a spin-1 combination and is nonzero only if the two spins form a spin-0. Then

$$\Phi_{ab} = \Phi\epsilon_{ab} \tag{3}$$

and Φ is the (scalar) order parameter. Show similarly that, in the He^3 case, $\Phi_{m,ab}$ vanishes if (ab) forms a spin-0 combination and is nonzero only if the two spins form a spin-1.

It is most convenient to write the order parameter as

$$\Phi_{ik} \tag{4}$$

where i is a 3-dimensional vector index for the orbital angular momentum and k is a 3-dimensional vector index for the spin-1 spin orientation. Please use this notation in the rest of the question. In the following, assume that the Gibbs free energy is invariant under separate orbital and spin rotations,

$$\Phi_{ik} \to R_{ij}\Phi_{j\ell}R'_{k\ell} \tag{5}$$

$i, j, k, \ell = 1, 2, 3$ where R is a 3×3 rotation matrix for the orbital angular momentum and R' is a 3×3 rotation matrix for the spin angular momentum. In the following, I will often write Φ as a matrix and use matrix multiplication to sum over indices. For example,

$$\sum_{ik} (\Phi_{ik})^* \Phi_{ik} = \mathrm{tr}[\Phi^\dagger \Phi] \tag{6}$$

Then the above transformation is

$$\Phi \to R\Phi(R')^T \tag{7}$$

and this is a symmetry of the Hamiltonia independently for R and R'.

(b) Show that the most general form of the Landau free energy is

$$G = \int d^3x \Big\{ \mathrm{tr}[(\vec{\nabla}\Phi^\dagger)(\vec{\nabla}\Phi)] - A(T)\mathrm{tr}[\Phi^\dagger\Phi] + B(\mathrm{tr}[\Phi^\dagger\Phi])^2 + C\mathrm{tr}[\Phi^\dagger\Phi\Phi^\dagger\Phi] \Big\} \,. \tag{8}$$

(c) To find the thermodynamic ground states, it is convenient to use the symmetries (5) to diagonalize Φ. This allows us to write

$$\Phi = R \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} (R')^T \tag{9}$$

where, if Φ is a constant in space, the Gibbs free energy does not depend on R, R'. Show that, for $C = 0$, the Gibbs free energy is stable as $\Phi \to \infty$ in any direction if and only if $B \geq 0$. Find the stability condition if B and C are both nonzero.

(d) Write $A = a(T_c - T)$. Show that the minimum of G is at a nonzero value if $T < T_c$. Find the minima of G for $C = 0$. In this case, the minima are highly degenerate.

(e) Find the minima of G for $B > 0$, $C > 0$. There are several distinct solutions to the variational equations for G. You should find the global minimum. The solution in this case is called the B-phase of He^3.

(f) Find the minima of G for $B > 0$, $C < 0$. There are several distinct solutions to the variational equations for G. You should find the global minimum. The solution in this case is called the A-phase of He^3.

(g) In each of the three cases, consider fluctuations in the direction $\Phi \to (1+\epsilon(x))\,\langle\Phi\rangle$ and compute the correlation length for $\epsilon(x)$.

(h) Consider now a putting our idealized superfluid He^3 between two metal walls parallel to the (x, y) plane, with separation much greater than the correlation length. The walls impose a boundary condition that the orbital angular momentum vector of the Cooper pairs must be perpendicular to the walls. This breaks the R symmetry. In the A-phase, find the configuration $\Phi(\vec{x})$ that minimizes G.

(i) It is not so easy to solve this problem for the parameters that give the B-phase. Solve this easier problem: If Φ obeys the boundary condition that it is 0 on the walls, find the configuration $\Phi(\vec{x})$ that minimizes G.

(j) Sketch the form of the configuration $\Phi(\vec{x})$ that minimizes G under the boundary condition of part (h) for parameters for which the bulk system is in the B phase, and exhibit its form near the wall and as it approaches its asymptotic value in the interior.

Our He^3 is much simpler than real superfluid He^3. For a taste of the real features that emerge with spin-orbit coupling, see the 2003 Nobel Prize lecture by Anthony Leggett. Leggett, quite appropriately, shared this Nobel Prize with Vitaly Ginzburg (Landau's co-author on the Landau-Ginzburg theory of superconductivity) and Alexei Abrikosov (of the flux tubes).