Physics 212 – Quiz #1

(issued Monday, October 18; due Friday, October 22)

I will use the quizzes and the final in this course to assign grades, keeping in mind that anyone who hands in all of the problem sets is assured a reasonable grade in the course. Because these quizzes will be graded, please abide by these rules:

- The quizzes are open-book. You may use any reference resources that you find. You may use your mathematical software such as MatLab or Mathematica. However, please do not collaborate with other students or ask help from other people—except that, if you have any questions about the quiz, please feel free to email me (mpeskin@slac.stanford.edu).
- The quizzes are posted at the course web site:

  `https://www.slac.stanford.edu/~mpeskin/Physics212/`

  This is the cover page. Please hand in your solution (upload to Gradescope) within 24 hours of the time that you turn the page and begin to solve the quiz.
- Please write on your solution: "I acknowledge the Stanford Honor Code." and sign it.
- The quizzes should be turned by Friday of the week in which they are issued. If this is a problem for you, please email me.

Each quiz will be worth 25 points. Partial credit will be given.

1. Consider the XY model on a 1-dimensional lattice

$$\mathcal{H} = -J \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} \ . \tag{1}$$

with periodic boundary conditions, in zero external magnetic field. The XY spin is a 2-component unit vector $\vec{S}_i = (\cos\phi_i, \sin\phi_i)$.

   (a) Write the transfer matrix for this model. On what Hilbert space does it act? The "matrix" here is a linear integral operator on this Hilbert space.

   (b) The eigenvector associated with the largest eigenvalue of the transfer matrix is a function of ϕ that is constant in ϕ:

   $$\psi_0(\phi) = (\text{const}) \tag{2}$$

   Find the corresponding eigenvalue. (Bessel functions appear; see the mathematical hint below.)

   (c) Compute the free energy for this model for $N \to \infty$. Find its high- and low-temperature limits.

   (d) Find the other eigenvectors and eigenvalues of the transfer matrix.

   (e) Compute the spin-spin correlation function

   $$\left\langle \vec{S}_I \cdot \vec{S}_J \right\rangle \tag{3}$$

   in terms of the eigenvalues of the transfer matrix.

   (f) Show, at least in the limits of high and low temperature, that the eigenvalue of ψ_0 actually is larger than any other eigenvalue. Show that this implies that there is no critical point at any finite temperature.

   (g) Work out the leading high- and low-temperature approximations for the correlation length.

   (h) If your mathematical software will plot Bessel functions, make a plot of the correlation length as a function of temperature for $J = 1$. It will be interesting for you to see this curve. if this is not possible for you, just make a sketch.

The integral

$$I_n(z) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \, e^{z\cos\varphi} \, \cos(n\varphi) \ , \tag{4}$$

for n an integer, will be useful to you. $I_n(z)$ is the modified Bessel function. You can easily look up its properties.