Physics 212 – Statistical Mechanics

Quiz #3 - Solution

1. (a) Following the analysis in class, at the Gaussian fixed point, any term

$$\int d^d x \; S^4$$

quartic in fields transforms under a change of scale as

$$\rightarrow \int d^d x \; \lambda^d \, \lambda^{4\left(\frac{d-2}{2}\right)} S^4 = \int d^d x \, S^4 \cdot \lambda^{d-4}$$

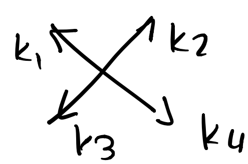
In $d = 4$, this term is scale-invariant before we consider interactions and thus is marginal.

(b) First study the correlation function

$$\langle S_1(x_1) \, S_1(x_2) \, S_1(x_3) \, S_1(x_4) \rangle$$

as a perturbation series in b. At order b^0, this has no connected part. The first contribution to the connected part comes in order b^1. Taking k_1 to be the Fourier transform variable for x_1, etc., this contribution is

$$\langle S_1 S_1 S_1 S_1 \rangle = \text{F.T.} \Big\{$$



$$\Big\} \leftarrow b \int S_1^4$$

There are 4! ways to connect the four external fields to the 4 fields in the b term, to the value of this diagram is

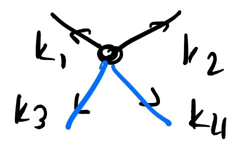
$$4! \left(-\frac{b}{4}\right) \prod_i \frac{1}{k_i^2} = (-6b) \prod_i \frac{1}{k_i^2}$$

Similarly, we will want to study the correlation function

$$\langle S_1(x_1)\, S_1(x_2)\, S_2(x_3)\, S_2(x_4) \rangle$$

This correlation function receives it first connected contribution in order c^1. It comes from the diagram

$$\langle S_1 S_1 S_2 S_2 \rangle = \text{F.T.}\left\{ \right.$$

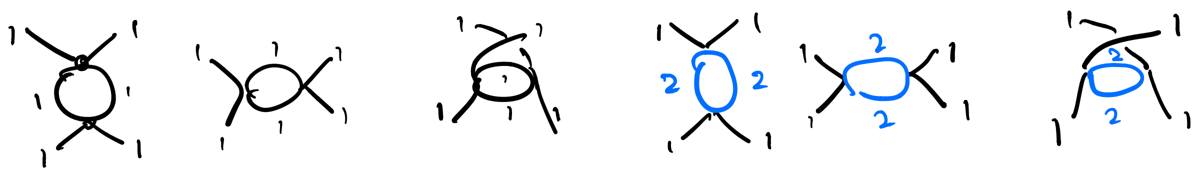


$$\left. \right\} \leftarrow c\int S_1^2 S_2^2$$

Black lines denote contractions of S_1's. Blue lines denote contractions of S_2's. Noting that contracting 2 S_1's and 2 S_2's to the c vertex can be done in $2 \cdot 2 = 4$ different ways, the value of this diagram is

$$4 \cdot \left(-\frac{c}{2}\right) \prod_i \frac{1}{k_i^2} = (-2c) \prod_i \frac{1}{k_i^2}$$

Let's now study the perturbative corrections to the first correlation function. There are 6 diagrams that contribute in order $(b, c)^2$. These are



The last 3 diagrams have a loop formed by contractions of S_2 between two c verticees.

The first three diagrams were computed in class. The first diagram contributes

$$+\, 18 b^2 \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{\ell^2 (\ell + q)^2}$$

times the factor $\prod(1/k_i^2)$, which is common to all diagrams. I will drop this factor from here on. We can obtain the overall factor in several ways, but here is a simple one: The natural factor would be $(-6b)^2$, counting 4! contractions with each vertex. In the expansion of the exponential, there is a factor of 1/2! at second order, but this is compensated by a factor 2 second we can contract $S_1(x_1)$ with either of the two S_1^4 terms at this order. Finally, this overcounts by a factor 2, since there is only 1 way to contract S_1^2 in the first vertex with S_1^2 in the second vertex. The final result is

$$2 \cdot \frac{1}{2!} \cdot (-6b)^2 \cdot \frac{1}{2} = +18b^2$$

which agrees with the above. After integrating over the momentum shell $\pi/\lambda a < |\ell| < \pi/a$ that we are integrating out, we found

$$+18b^2 \frac{1}{(4\pi)^2} \log \lambda^2$$

for the final value of this diagram. The second and third diagrams contribute this same amount.

The fourth diagram contributes

$$\frac{1}{2}(-2c)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 (\ell+q)^2}$$

Note that I have divided by the same factor of 2 as in the previous paragraph. Performing the integral, we find

$$+2c^2 \frac{1}{(4\pi)^2} \log \lambda^2$$

The fifth and sixth diagrams give the same contribution.

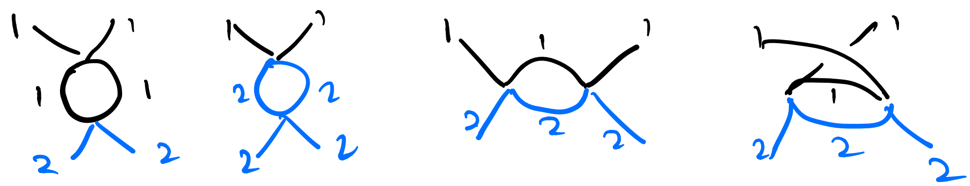
In all, the second-order contribution is

$$(18b \cdot 3 + 2c^2 \cdot 3) \frac{1}{(4\pi)^2} \log \lambda^2$$

Comparing to the factor $(-6b)$ in the order b^1 result and receiving a factor 2 by differentiating the logarithm with respect to λ, we find the RG equation

$$\ell \frac{d}{d\ell} b = -\frac{9}{8\pi^2} b^2 - \frac{1}{8\pi^2} c^2$$

Now study the perturbative corrections to the second correlation function. There are 4 diagrams



The value of the first diagram is

$$\frac{(-6b)(-2c)}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell+q)^2} = 6bc\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell+q)^2}$$

Again, we need to divide by the factor 2 for the contraction of S_1^2 in the top vertex with S_1^2 in the bottom vertex. The second diagram gives the same contribution. The third diagram gives

$$(-2c)^2\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell+q)^2} = 4c^2\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2(\ell+q)^2}$$

Note that there is no factor of 2 amiss here. The fourth diagram gives the same contribution. In all, after integrating over the momentum shell, the full contribution is

$$(12\,bc + 8c^2)\frac{1}{(4\pi)^2}\log\lambda^2$$

This is to be compared to the factor $(-2c)$ from the order c^1 result. The RG equation is then

$$l \frac{d}{dl} c = -\frac{6}{8\pi^2} bc - \frac{4}{8\pi^2} c^2$$

Let me repeat the two RG equations:

$$l \frac{d}{dl} b = -\frac{9}{8\pi^2} b^2 - \frac{1}{8\pi^2} c^2$$

$$l \frac{d}{dl} c = -\frac{6}{8\pi^2} bc - \frac{4}{8\pi^2} c^2$$

You see that setting $c = 0$ reproduces the result from class. Setting $b = c$, both equations reduce to

$$l \frac{d}{dl} b = -\frac{10}{8\pi^2} b^2$$

which the RG equation for the $n = 2$ Landau theory discussed in class.

(c) In $d = 4 - \epsilon$, we add a linear term in b, c expressing the scaling shown in part (a). The RG equations are now

$$l \frac{d}{dl} b = \epsilon b - \frac{9}{8\pi^2} b^2 - \frac{1}{8\pi^2} c^2$$

$$l \frac{d}{dl} c = \epsilon c - \frac{6}{8\pi^2} bc - \frac{4}{8\pi^2} c^2$$

(d) The fixed points of these equations are given by

$$\epsilon b = \frac{9}{8\pi^2} b^2 + \frac{1}{8\pi^2} c^2$$

$$\epsilon c = \frac{6}{8\pi^2} bc + \frac{4}{8\pi^2} c^2$$

To solve these equations, set $r = c/b$. Then, at a fixed point,

$$8\pi^2 \epsilon r b = r(9+r^2)b^2 = (6r+4r^2)b^2$$

This gives an equation for r

$$r^3 - 4r^2 + 3r = 0$$

which factors into

$$r(r-1)(r-3) = 0$$

So there are fixed points with $r = 0, 1, 3$,

$$b_* = \frac{8\pi^2}{9}\epsilon \qquad c_* = 0$$

$$b_* = \frac{8\pi^2}{10}\epsilon \qquad c_* = \frac{8\pi^2}{10}\epsilon$$

$$b_* = \frac{8\pi^2}{18}\epsilon \qquad c_* = \frac{8\pi^2}{6}\epsilon$$

(e) To sketch the flows, we need to find the stability of each fixed point. To do this, we linearize the RG equations about each fixed point.

Consider first the point at $c = 0$. Write

$$b = \frac{8\pi^2}{9}\epsilon + \beta \qquad c = \gamma$$

The linearized equations are

$$\ell\frac{d}{d\ell}\beta = \epsilon\left(\frac{8\pi^2}{9}+\beta\right) - \frac{9}{8\pi^2}\left(\frac{8\pi^2}{9}+\beta\right)^2 + \cdots$$

$$\ell\frac{d}{d\ell}\gamma = \epsilon\gamma - \frac{6}{8\pi^2}\left(\frac{8\pi^2}{9}+\gamma\right)^2 + \cdots$$

that is,

$$\ell\frac{d}{d\ell}\beta = -\epsilon\beta$$

$$\ell\frac{d}{d\ell}\gamma = +\frac{1}{3}\epsilon\gamma$$

Then the b direction is attractive, but the c direction is repulsive. Next consider the point at $b = c$. Write

$$b = \left(\frac{8\pi^2}{10}+\beta\right) \qquad c = \frac{8\pi^2}{10}+\gamma$$

The linearized equations are

$$\ell\frac{d}{d\ell}\beta = \epsilon\left(\frac{8\pi^2}{10}+\beta\right) - \frac{9}{8\pi^2}\left(\frac{8\pi^2}{10}+\beta\right)^2 - \frac{1}{8\pi^2}\left(\frac{8\pi^2}{10}+\gamma\right)^2$$

$$\ell\frac{d}{d\ell}\gamma = \epsilon\left(\frac{8\pi^2}{10}+\gamma\right) - \frac{6}{8\pi^2}\left(\frac{8\pi^2}{10}+\beta\right)\left(\frac{8\pi^2}{10}+\gamma\right) - \frac{4}{8\pi^2}\left(\frac{8\pi^2}{10}+\gamma\right)^2$$

This has the form

$$\ell\frac{d}{d\ell}\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = -\mathcal{M}\begin{pmatrix}\beta\\ \gamma\end{pmatrix}$$

with

$$\mathcal{M} = \begin{pmatrix} 4/5\,\epsilon & 1/5\,\epsilon \\ 3/5\,\epsilon & 2/5\,\epsilon \end{pmatrix}$$

Positive eigenvalues of $\mathcal{M}$ correspond to attractive directions. One eigenvector of this matrix is easily found

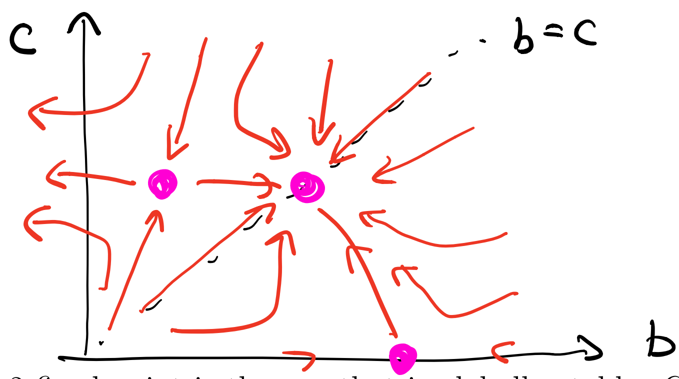
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \quad \lambda = +\epsilon$$

This is just the stability of the fixed point in the $n = 2$ Landau theory. The characteristic equation that gives the eigenvalues of the matrix is

$$0 = \lambda^2 - \frac{6}{5}\epsilon\lambda + \frac{1}{5}\epsilon^2 = (\lambda - \epsilon)\left(\lambda - \frac{\epsilon}{5}\right)$$

so the other eigenvalue is also positive, making this fixed point stable in both directions.

We can fill in the rest of the diagram by continuity



The $n = 2$ fixed point is the one that is globally stable. General models in the basin of attraction of the fixed point are drawn into the fixed point when T is very close to T_c. Thus, for a large part of the (b, c) plane, models that are not $SO(2)$-symmetric have the critical exponents of the $n = 2$ Landau theory.

(f) Notice that, if the initial value of b is much less than the value of c, the RG equation drives b negative. Then the effective potential free energy of the theory is unstable with respect to generating a very large value of $\langle S_1 \rangle$ or $\langle S_2 \rangle$. In this region, there is no continuous phase transition. Instead the transition to the ordered state will be discontinuous (first-order).