Physics 212 Quiz #2

Solutions

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a.) I expect you to know from quantum mechanics that the spin 0 combination of two spin-$\frac{1}{2}$'s is antisymmetric

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

and the spin 1 combination of two spin-$\frac{1}{2}$'s is symmetric

$$|\uparrow\uparrow\rangle \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \qquad |\downarrow\downarrow\rangle$$

so this problem boils down to finding the symmetry of Φ_{ab}.

For the superconductor

$$\Phi_{ab} = \int \frac{d^3k}{(2\pi)^3}\, a_{\vec{k},a}\, a_{-kb}$$

change variables $\vec{k} \to -\vec{k}$

$$= \int \frac{d^3k}{(2\pi)^3}\, a_{-ka}\, a_{kb}$$

but a_{ka} and a_{-kb} anticommute

$$= -\int \frac{d^3k}{(2\pi)^3}\, a_{kb}\, a_{-ka} = -\Phi_{ba}$$

so Φ_{ab} is antisymmetric in a,b and so is the spin-0 combination.

Similarly, in the He^3 case

$$\Phi_{m,ab} = \int \frac{d^3k}{(2\pi)^3} Y_{1m}(\hat{k})\, a_{ka}\, a_{-kb}$$

change variables $\vec{k} \to -\vec{k}$ and use $Y_{1m}(-\hat{k}) = -Y_{1m}(\hat{k})$

$$= \int \frac{d^3k}{(2\pi)^3} \left(-Y_{1m}(\hat{k})\right) a_{-ka}\, a_{kb}$$

$$= +\int \frac{d^3k}{(2\pi)^3} Y_{1m}(\hat{k})\, a_{kb}\, a_{-ka} = \Phi_{m,ba}$$

so $\Phi_{m,ab}$ involves the <u>symmetric</u>, therefore spin-1, combination

[In case you are curious, the connection between the representation (ab) and the representation in terms of a 3-vector index is

$$|k\rangle = i(\sigma^k \varepsilon)_{ab}\ |ab\rangle$$

where σ^k is a Pauli matrix and $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that

$$(-i\sigma^1\varepsilon) = \begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix} \qquad (i\sigma^2\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (i\sigma^3\varepsilon) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

are <u>symmetric</u> matrices. This information is not needed to do the rest of the quiz.]

b.) The Landau free energy is a polynomial in Φ, Φ^+ that is invariant to the symmetries R, R' with

$$R^+ = R^T = R^{-1} \quad \text{so} \quad R^T R = 1$$

Then since

$$\Phi \to R\,\Phi\,R'^T \qquad \Phi^+ \to R'\,\Phi^+\,R^T$$

the invariants take the form $\quad tr[\Phi^+\Phi],\ tr[\Phi^+\Phi\Phi^+\Phi]$ etc.

up to quartic order the only invariants are

$$\mathrm{tr}[\Phi^\dagger\Phi] \qquad (\mathrm{tr}\,\Phi^\dagger\Phi)^2 \qquad \mathrm{tr}[\Phi^\dagger\Phi\Phi^\dagger\Phi]$$

c.) Write $\Phi = R \begin{pmatrix} X & & \\ & Y & \\ & & Z \end{pmatrix} (R')^T$ and assuming that Φ is constant in space, G takes the form:

$$G = (\text{Volume}) \cdot \left[-A\,(X^2+Y^2+Z^2) + B\,(X^2+Y^2+Z^2)^2 + C\,(X^4+Y^4+Z^4) \right]$$

If $C = 0$, the B term dominates the A term in any direction. So if G is not to go to $-\infty$ as $(X,Y,Z) \to \infty$, we need $\underline{B \geq 0}$

Similarly if $B = 0$ we need $\overline{C \geq 0}$

If both B and C are nonzero the system is stable if both are positive. If one is negative, we require

if $B > 0 \quad C < 0 \qquad B \geq |C|$

if $B < 0 \quad C > 0 \qquad C > |B|$

d) If $A = a(T_c - T)$ and $T < T_c$ then near the symmetric point $X = Y = Z$ G behaves as

$$G \sim -|A|\,(X^2+Y^2+Z^2)$$

and so this point is an un<u>stable</u> maximum. The minimum of G must be at a point where (X,Y,Z) is nonzero.

If $C = 0$ the minimization problem is

$$G/V = -A(X^2+Y^2+Z^2) + B(X^2+Y^2+Z^2)^2$$

so there is a <u>sphere</u> of minima with

$$(X^2+Y^2+Z^2) = R^2$$

where R is determined by

$$0 = \frac{\partial}{\partial R}\left(-A(R^2) + B(R^2)^2\right)$$

$$= -2AR + 4BR^3$$

so $|(X,Y,Z)| = R$ $\quad R = \left[\frac{A}{2B}\right]^{\frac{1}{2}}$

e.) Now consider $B>0, C>0$. We need to minimize

$$G/V = -A(X^2+Y^2+Z^2) + B(X^2+Y^2+Z^2)^2 + C(X^4+Y^4+Z^4)$$

$$0 = \frac{\partial}{\partial X}(G/V) = -2AX + 4BX(X^2+Y^2+Z^2) + 4CX^3$$

$$0 = \frac{\partial}{\partial Y}(G/V) = -2AY + 4BY(X^2+Y^2+Z^2) + 4CY^3$$

$$0 = \frac{\partial}{\partial Z}(G/V) = -2AZ + 4BZ(X^2+Y^2+Z^2) + 4CZ^3$$

so each of X, Y, Z solves

$$A = 2B(X^2+Y^2+Z^2) + 2C\,W^2 \qquad W = X, Y, Z$$

or $\quad W = 0$

up to $X \to -X$, $Y \to -Y$, $Z \to -Z$, there are 3 solutions:

① one variable is nonzero, say $X \neq 0$ $Y, Z = 0$

$$A = (2B + 2C)\,X^2$$

$$X = \left(\frac{A}{2B+2C}\right)^{\frac{1}{2}}$$

② two variables are nonzero say $X, Y \neq 0$ $Z = 0$

$$X = Y = \left(\frac{A}{4B+2C}\right)^{\frac{1}{2}}$$

③ all three variables are nonzero

$$X = Y = Z = \left(\frac{A}{6B+2C}\right)^{\frac{1}{2}}$$

Work out G/V in these 3 cases:

① $$G/V = -A\,\frac{A}{2(B+C)} + (B+C)\left[\frac{A}{2(B+C)}\right]^2$$

$$= -\frac{1}{4}\left(\frac{A^2}{B+C}\right)$$

② $$G/V = -A\,\frac{A}{2(2B+C)}\cdot 2 + B\left[2\,\frac{A}{2(2B+C)}\right]^2 + C\;2\cdot\left[\frac{A}{2(2B+C)}\right]^2$$

$$= -\frac{A^2}{2B+C} + \frac{(2B+C)}{2}\,\frac{A^2}{(2B+C)^2}$$

$$= -\frac{1}{2}\,\frac{A^2}{(2B+C)}$$

③ $$G/V = -A\,\frac{A}{2(3B+C)}\cdot 3 + B\left[3\cdot\frac{A}{2(3B+C)}\right]^2 + C\cdot 3\left[\frac{A}{2(3B+C)}\right]^2$$

$$= -\frac{3}{2}\,\frac{A^2}{(3B+C)} + (3B+C)\,\frac{3}{4}\,\frac{A^2}{(3B+C)^2}$$

$$= -\frac{3}{4}\,\frac{A^2}{(3B+C)}$$

Then results are

① ② ③

$$-\frac{1}{4}\left(\frac{A^2}{B+C}\right) > -\frac{1}{4}\,\frac{A^2}{(B+C/2)} > -\frac{1}{4}\,\frac{A^2}{(B+C/3)}$$

So ③ gives the absolute minimization. The general form is

$$\Phi = R\begin{pmatrix}\pm\Phi_0 & & \\ & \pm\Phi_0 & \\ & & \pm\Phi_0\end{pmatrix}(R')^T$$

where $$\Phi_0 = \left[\frac{A}{2(B+C/3)}\right]^{1/2}$$

This is the B-phase. To work with this, I'll just write

$$\Phi = \begin{pmatrix} \Phi_0 & & \\ & \Phi_0 & \\ & & \Phi_0 \end{pmatrix}$$

f.) For $C < 0$, the same formulae hold; but the ordering is different

$$\underset{①}{-\frac{1}{4}\frac{A^2}{(B-|C|)}} < \underset{②}{-\frac{1}{4}\frac{A^2}{(B-|C|/2)}} < \underset{③}{-\frac{1}{4}\frac{A^2}{(B-|C|/3)}}$$

so the general form is

$$\Phi = R\begin{pmatrix} \Phi_0 & & \\ & 0 & \\ & & 0 \end{pmatrix}(R')^T \qquad \Phi_0 = \left[\frac{A}{2(B-|C|)}\right]^{1/2}$$

This the A phase.

In our simplified model, the direction of the spin and orbital angular momentum vectors are arbitrary and independent. In real superfluid He^3, the angle between these vectors is fixed by spin-orbit interaction.

g.) Vary Φ in the direction of the order parameter.

$$\Phi \rightarrow \langle\Phi\rangle(1+\varepsilon(x))$$

$$\mathrm{tr}\left[\vec{\nabla}\Phi^\dagger \vec{\nabla}\Phi\right] \rightarrow \mathrm{tr}(\langle\Phi^\dagger\rangle\langle\Phi\rangle)(\vec{\nabla}\varepsilon)^2$$

$$\mathrm{tr}\left[\Phi^\dagger\Phi\right] \rightarrow \mathrm{tr}\left(\langle\Phi^\dagger\rangle\langle\Phi\rangle\right)(1+\varepsilon)^2$$

$$\left(\mathrm{tr}\left(\Phi^\dagger\Phi\right)\right)^2 \rightarrow \langle\mathrm{tr}(\Phi^\dagger\Phi)\rangle^2(1+\varepsilon)^4$$

$$\mathrm{tr}\left(\Phi^\dagger\Phi\Phi^\dagger\Phi\right) \rightarrow \mathrm{tr}\,\langle\Phi^\dagger\Phi\Phi^\dagger\Phi\rangle\;(1+\varepsilon^4)$$

Analyze ~~first~~ the case $B>0$, $C>0$

$$\langle \Phi \rangle = \begin{pmatrix} \Phi_0 & & \\ & \Phi_0 & \\ & & \Phi_0 \end{pmatrix} \qquad \Phi_0 = \left[\frac{A}{2(3B+C)}\right]^{1/2}$$

using this expectation value

$$\mathrm{tr}(\Phi^\dagger\Phi) = 3\Phi_0^2 \qquad (\mathrm{tr}\,\Phi^\dagger\Phi)^2 = 9\Phi_0^4 \qquad \mathrm{tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi) = 3\Phi_0^4$$

then

$$G \to \int d^3x \Big\{ (3\Phi_0^2)(\vec\nabla\varepsilon)^2 - A(3\Phi_0^2)(1+2\varepsilon+\varepsilon^2)$$
$$+ B(9\Phi_0^4)(1+4\varepsilon+6\varepsilon^2+\cdots) + C(3\Phi_0^4)(1+4\varepsilon+6\varepsilon^4)\Big\}$$

the ε terms are

$$\varepsilon\cdot(3\Phi_0^2)(-2A + 4B\,3\Phi_0^4 + 4C\,\Phi_0^4)$$
$$= \varepsilon\cdot 2\Phi_0^2\left(-2A + 4\cdot(3B+C)\frac{A}{2(3B+C)}\right)$$
$$= 0 \quad \text{as required at the minimum.}$$

the ε^2 terms in the potential are

$$3\Phi_0^2(-A + 6(3B+C)\Phi_0^2)\cdot\varepsilon^2$$
$$= 3\Phi_0^2\left(-A + 6(3B+C)\frac{A}{2(3B+C)}\right)\varepsilon^2$$
$$= 3\Phi_0^2\left(-\frac{2A}{(\ )}\right)\varepsilon^2$$

so

$$G \to \int d^3x\,(3\Phi_0^2)\left[\varepsilon\left[-\nabla^2 + 2A\right]\varepsilon\right]$$

then the Green's funct. for ε is

$$\langle \varepsilon(x)\, \varepsilon(0) \rangle \sim (\text{const}) \cdot \frac{1}{|x|}\, e^{-(2A)^{1/2}|x|}$$

and the correlation length is

$$\xi = \frac{1}{(2A)^{1/2}}$$

If all is true and just in the world, we ought to get the same answer for $B > 0$, $C < 0$. Let's check: For the expectation values

$$\operatorname{tr} \Phi^\dagger \Phi = \Phi_0^2 \qquad \Phi_0 = \left[\frac{A}{2(B-|C|)}\right]^{1/2}$$

$$(\operatorname{tr} \Phi^\dagger \Phi)^2 = \Phi_0^4 \qquad \operatorname{tr}[\Phi^\dagger \Phi \Phi^\dagger \Phi] = \Phi_0^4$$

$$G \to \int d^3x \left\{ \Phi_0^2 |\vec{\nabla}\varepsilon|^2 - A\, \Phi_0^2 (1 + 2\varepsilon + \varepsilon^2) \right.$$
$$\left. + B\, \Phi_0^4 (1 + 4\varepsilon + 6\varepsilon^2) + C\, \Phi_0^4 (1 + 4\varepsilon + 6\varepsilon^2) + \cdots \right\}$$

the ε term is

$$\Phi_0^2 \cdot 2\varepsilon \left(-A + 2B\Phi_0^2 - |C| \cdot 2\Phi_0^2 \right)$$

$$= 2\varepsilon \Phi_0^2 \left(-A + 2(B-|C|) \cdot \frac{A}{2(B-|C|)} \right) = 0$$

as required

the ε^2 term is

$$\Phi_0^2\, \varepsilon^2 \left(-A + 6(B-|C|)\Phi_0^2 \right)$$

$$= \varepsilon^2 \Phi_0^2 \left(-A + 6(B-|C|) \frac{A}{2(B-|C|)} \right)$$

$$= \varepsilon^2 \Phi_0^2 (-A + 3A) = +2A \Phi_0^2 \varepsilon^2$$

so

$$G \to \int d^3x \, \Phi_0^2 \, \varepsilon (-\nabla^2 + 2A) \varepsilon \qquad \text{as before}$$

and so

$$\xi = \frac{1}{(2A)^{1/2}}$$

the two solutions become equivalent at $C = 0$ so

$$\xi = 1/(2A)^{1/2} \qquad \text{in all cases.}$$

h.) We are in the A-phase. The boundary condition is

$$\Phi_{ik} = 0 \quad \text{for } i = x, y \text{ at the wall.}$$

so we can solve the problem with the uniform solution

$$\Phi_{ik} = \begin{pmatrix} 0 & 0 & \Phi_0 \end{pmatrix}$$

or

$$\Phi = \begin{pmatrix} 0 & 0 & \Phi_0 \end{pmatrix} (R')^T \qquad \text{for any } R'$$

independent of $\underline{z}$.

ii.) Now consider the B-phase, $B>0$ $C>0$

with the boundary condition $\Phi = 0$ on the wall

Look for a solution

$$\Phi = \begin{pmatrix} \Phi_0 & & \\ & \Phi_0 & \\ & & \Phi_0 \end{pmatrix} f(z) \qquad \Phi_0^2 = \frac{A}{2(3B+C)}$$

where $f(z) = 0$ at $z=0$ and $f(z) \to 1$ as $z \to \infty$

$$G = \int d^3x \left\{ 3\Phi_0^2 (\vec{\nabla} f)^2 - A\,(3\Phi_0^2)\, f^2 \right.$$
$$\left. + \left(B\, 9\Phi_0^4 + C\, 3\Phi_0^4\right) f^4 \right\}$$

vary with respect to $f(z)$

$$\frac{\delta G}{\delta f(z)} = 0 = 2(3\Phi_0^2)\left[-\nabla^2 f - Af + \underbrace{2(3B+C)\Phi_0^2}_{A} f^3\right]$$

so

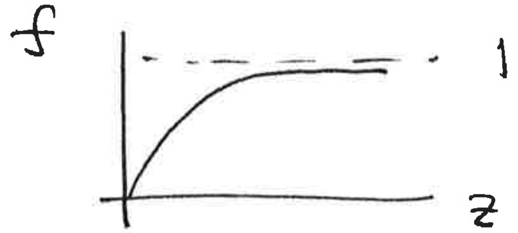
$$-\nabla^2 f - Af + Af^3 = 0$$

we encountered this equation in our study of the domain wall. The solution is

$$f(z) = m_0 \tanh(z/2\xi)$$

where m_0 is the minimum of the potential or the solution for constant f: $m_0 = 1$, and ξ is the correlation length $\xi = 1/(2A)^{1/2}$

to satisfy the boundary condition, we need ½ of this solution:



so

$$\Phi(x) = \begin{pmatrix} \Phi_0 & & \\ & \Phi_0 & \\ & & \Phi_0 \end{pmatrix} \tanh\left(\frac{A}{\sqrt{2}} z\right)$$

j.) The correct boundary condition for the B phase is still $\Phi_{ik} = 0$ for $i = x, y$ only.

So the true solution will look something like

$$\Phi_0 = \begin{pmatrix} \Phi_0 \, g(z) & & \\ & \Phi_0 \, g(z) & \\ & & \Phi_0 \, h(z) \end{pmatrix}$$

where