Physics 212 - Quiz #2
Solutions

1.) a.) The Landau theory must have the symmetry under 120° rotations of $\vec{m}$. In complex notation

$$m = m^1 + i m^2$$

this is

$$m(x) \to e^{2\pi i/3} m(x)$$

Polynomials invariant under this gravity are built from

$$|m|^2 \ , \quad m^3 \ , \ (m^*)^3$$

so the leading terms in the Landau theory are

no derivs: $\quad |m|^2 \quad |m|^4 \quad Re(m^3) \quad Re(m^6)$ etc

2 derivs: $\quad |\vec{\nabla} m|^2 \ \ldots$

up to quartic order in ∇ and m, quadratic order in ∇

$$G = \int d^3x \left\{ \tfrac{1}{2}(\vec{\nabla} m)^2 - A|m|^2 - \tfrac{1}{2}C\left[m^3 + (m^*)^3\right] + \frac{B}{4}|m|^4 \right\}$$

b.) To find the equilibrium states, set m = const and take $\partial / \partial m^*$ of G

$$-Am - \frac{3}{2} c (m^*)^2 + \frac{B}{2} m |m|^2 = 0$$

For small c, the easiest way to analyze this equation is to first ignore c and then add back the c term as a perturbation

With $c = 0$

$$-A + \frac{B}{2} |m|^2 = 0$$

so
$$m = \left[\frac{2A}{B}\right]^{1/2} e^{i\theta}$$

for any value of θ. The problem is U(1)-symmetric.

The c-term is

$$\Delta G / \text{Vol} = -\frac{c}{2} \left[(m^3) + (m^*)^3\right]$$

$$= -\frac{c}{2} m_0^3 \left[e^{3i\theta} + e^{-3i\theta}\right]$$

$$= -c\, m_0^3 \cos 3\theta$$

for $c > 0$, this is minimized at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

so for $c > 0$, the equilibrium states are

$$m = \left[\frac{2A}{B}\right]^{1/2} e^{i\theta} \qquad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

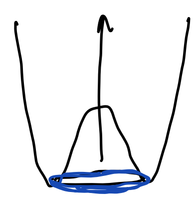
c.) By the same logic, for $c < 0$, the equilibrium states are at the minima of $\cos 3\theta$

$$m = \left[\frac{2A}{B}\right]^{1/2} e^{i\theta} \qquad \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

A moral to draw is that, for small c, the system is in a deep well at

$$|m| = \left[\frac{2A}{B}\right]^{1/2}$$

with a shallow potential that breaks the $U(1)$ symmetry



The ~~interest~~ dynamics of this theory involves θ only, with $|m|$ essentially frozen at $m_0 = \left[\frac{2A}{B}\right]^{1/2}$.

The effective theory of $\theta(x)$ is given by plugging

$$m(x) = m_0 e^{i\theta(x)}$$

into G. It is

$$G = \int d^3x \left\{ \frac{1}{2} m_0^2 |\vec{\nabla}\theta|^2 - c\, m_0^3 \cos 3\theta(x) \right\}$$

$$+ (\text{const})$$

I will use this formula to answer the rest of the Quiz.

d.) Expand about the minimum at $\theta = 0$. The other minima have the same properties, by symmetry.

$$\cos 3\theta \simeq 1 - \frac{1}{2}(3\theta)^2$$

$$G = \int d^3x \ \frac{1}{2} m_0^2 \left\{ (\vec{\nabla}\theta)^2 + c m_0 (3\theta)^2 \right\}$$

$$= \int d^3x \ \frac{1}{2} m_0^2 \left\{ \theta \left(-\nabla^2 + 9 c m_0\right) \theta \right\}$$

The Green's function for θ then satisfies

$$\left(-\nabla^2 + 9 c m_0\right) G(x,y) = T\delta(x-y)$$

so the correlation length of θ is

$$\xi^{-2} = 9 c m_0$$

so $\quad \xi \sim 1/\sqrt{c} \to \infty$ as $c \to \infty$

e.) The domain walls are field configurations that go from one minimum of G to another. Along the θ direction, the potential is

so there are possible domain walls

X: $\theta = 0$ to $\theta = \frac{2\pi}{3}$ $\quad$ $\bar{X}$ $\theta = \frac{2\pi}{3}$ to $\theta = 0$

Y: $\theta = \frac{2\pi}{3}$ to $\theta = \frac{4\pi}{3}$ $\quad$ $\bar{Y}$ $\theta = \frac{4\pi}{3}$ to $\theta = \frac{2\pi}{3}$

Z: $\theta = \frac{4\pi}{3}$ to $\theta = 0$ $\quad$ $\bar{Z}$ $\theta = 0$ to $\theta = \frac{4\pi}{3}$

The thickness of a domain wall should be of the order of the correlation length

$$\Delta x \sim \frac{1}{\sqrt{9 m_0 c}}$$

The domain wall tension is estimated as

$$\Delta G \cong (\text{Area}) \cdot \text{thickness} \cdot c m_0^3$$

$$= \text{Area} \cdot \frac{1}{\sqrt{9 m_0 c}} \, c m_0^2$$

so $\Delta G = \text{Area} \cdot T$

where $T \sim \frac{\sqrt{c}\, m_0^{3/2}}{3}$

f.) Varying the expression for G in part (d)

$$\frac{1}{2} m_0^2 \left(-\nabla^2 \theta + 3 c m_0 \sin 3\theta \right) = 0$$

$\theta(\vec{x})$ will be a function of x only, so

$$\frac{d^2}{dx^2} \theta = 3 c m_0 \sin 3\theta$$

It is convenient to let $\zeta = x/\xi = \sqrt{9 c m_0}\, x$.

Then
$$\frac{d^2}{d\zeta^2} \theta = \frac{1}{3} \sin 3\theta$$

This equation is called the "sine-Gordon equation".

We wish to find a solution with

$$\theta \to 0 \text{ as } \zeta \to -\infty \qquad \theta \to \frac{2\pi}{3} \text{ as } \zeta \to \infty$$

g.) Consider this problem as that for a particle in a potential and find the conserved energy. To do this, multiply by $\frac{d\theta}{d\zeta}$

and integrate

$$\frac{d\theta}{d\eta} \cdot \frac{d^2\theta}{d\eta^2} = \frac{1}{3} \sin 3\theta \ \frac{d\theta}{d\eta}$$

$$\frac{1}{2}\left(\frac{d\theta}{d\eta}\right)^2 = C - \frac{1}{9} \cos 3\theta$$

so

$$C = \frac{1}{2}\frac{d\theta}{d\eta} + \frac{1}{9} \cos 3\theta$$

as $x \to -\infty$, $\frac{d\theta}{d\eta} = 0$ $\cos 3\theta = 1$, so $C = \frac{1}{9}$

then $\left(\frac{d\theta}{d\eta}\right)^2 = \frac{2}{9}(1 - \cos 3\theta) = \frac{4}{9} \sin^2 \frac{3\theta}{2}$

$$\int \frac{d\theta}{\sin 3\theta/2} = \int d\eta \cdot \frac{2}{3}$$

$$\phi = \frac{3\theta}{2} \qquad d\theta = \frac{2}{3} d\phi$$

$$\int \frac{d\phi}{\sin\phi} = \int d\eta$$

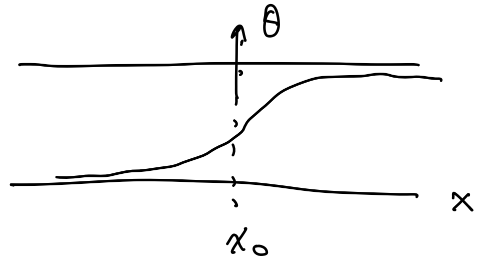
$$\log \tan \phi/2 = \eta - \eta_0$$

$$\phi = 2\tan^{-1}\left[e^{\eta - \eta_0}\right]$$

Finally
$$\theta(x) = \frac{4}{3}\tan^{-1}\left(\exp\left[\frac{x-x_0}{\xi}\right]\right)$$

so $x \to -\infty \quad \theta \to 0$

$x \to +\infty \quad \theta \to \frac{4}{3}\frac{\pi}{2} = \frac{2\pi}{3}$ ✓



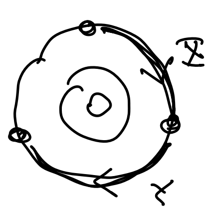
h.) There are two possible outcomes. I will give full credit for either one

1.) If the two domain walls come together with low energy, they will repel. Then the situation

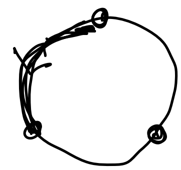
$$\Big|\ 0\ \Big\}\ \frac{2\pi}{3}\ \Big\}\ \frac{4\pi}{3}\ \Big|$$

will be stable.

2.) If the two domain wall come together with enough energy to get over the barrier



they can end up as a $\mathbb{Z}$ domain wall



which has the same topology as $\mathbb{Z}_3$.