Physics 212 – Statistical Mechanics

Quiz #2 - Solution

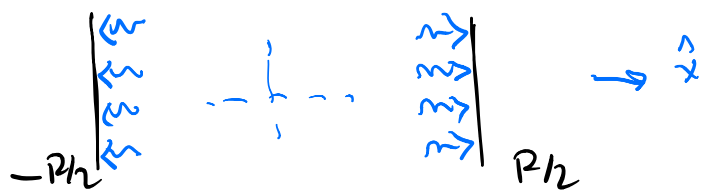
1. (a) For $T > T_c$, the free energy is approximately given by

$$G = \int d^3x \left\{ \frac{1}{2}(\nabla^i n^k)^2 + \frac{1}{2} a (T - T_c)(n^k)^2 \right\}$$

The variational equation for $\vec{n}(\vec{x})$ is

$$-\nabla^2 n^k + a(T - T_c) n^k = 0$$

The boundary conditions are



that is, $n^x = -N_0$ at $x = -R/2$, $n^x = +N_0$ at $x = R/2$, and $n^{y,z} = 0$ at both boundaries. We can take $\vec{n}$ to be a function of x only. Then

$$\frac{d^2}{dx^2} n^k = a(T - T_c) n^k$$

So it is consistent to have $n^y = n^z = 0$ everywhere, and

$$n^x = C \sinh [a(T - T_c)]^{\frac{1}{2}} x$$

where

$$C = \frac{N_0}{\sinh\left(a[T-T_c]^{1/2} R/2\right)}$$

Then, finally,

$$n^x = N_0 \frac{\sinh\left([a(T-T_c)]^{1/2} x\right)}{\sinh\left([a(T-T_c)]^{1/2} R/2\right)}$$

For future reference, let's compute the excess free energy associated with this solution. Above T_c, the ground state has zero free energy in the Landau model, so the excess is, using $\xi = [a(T-T_c)]^{-1/2}$,

$$G = (\text{Area}) \int_{-R/2}^{R/2} dx \left\{ \frac{1}{2}\left(\frac{d}{dx} n^x\right)^2 + \frac{1}{2} \xi^{-2} (n^x)^2 \right\}$$

$$= (\text{Area}) \int_{-R/2}^{R/2} dx \, \frac{N_0^2}{2} \xi^{-2} \left(\frac{\cosh^2 x/\xi}{\sinh^2 R/2\xi} + \frac{\sinh^2 x/\xi}{\cosh^2 R/2\xi} \right)$$

For $R \gg \xi$, the free energy is dominated by the region where the sinh is large, that is, the region within ξ of the boundaries at $x = \pm R/2$.

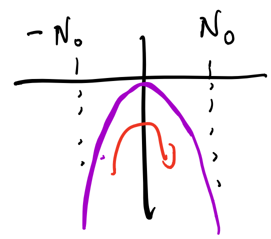
$$G \cong (\text{Area}) \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{N_0^2}{\xi} \int_0^{R/2} dx/\xi \, e^{R/2\xi - x/\xi}$$

or finally

$$G \cong (\text{Area}) \cdot \frac{2}{\xi} N_0^2$$

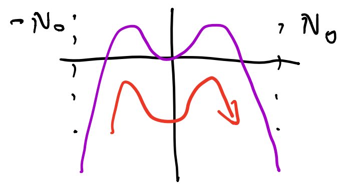
(b) In the mechanical analogy, the potential is (-1) times the potential terms in the variational equation. In this problem

$$V = -\frac{a}{2}(T-T_c)(n^x)^2$$



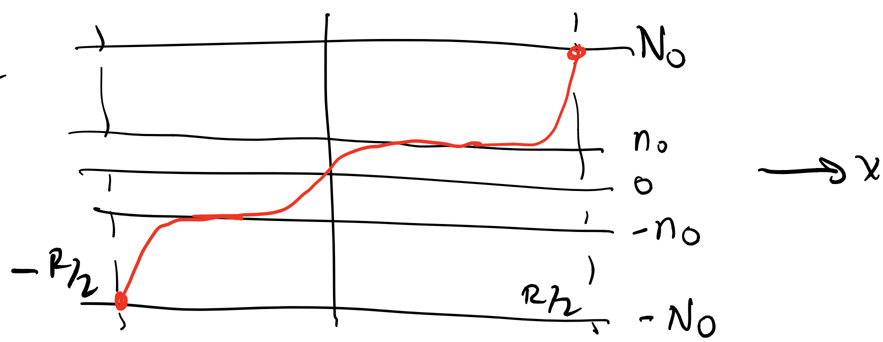
Then the solution in part (a) corresponds to the situation in which the initial particle, at time $-N_0$, is given just barely enough kinetic energy to go up to the top of the hill. After spending a long time there, it falls down to its original level at time $+N_0$.

The situation for $T > T_c$ is basically the same if we include the b term. The situation changes only when $T < T_c$. Then the potential has the form



Now $\xi = [2a(T - T_c)]^{-1/2}$. For $R/2 \gg \xi$, the motion consists of five parts, an ascent to the first peak, a period of waiting at the top of the first peak, a transition between the peaks that has the form of the domain wall discussed in class, a period of waiting at the second peak, and the descent to the original level at time $+N_0$.

$$n_0 = \left[\frac{a(T-T_c)}{b}\right]^{\frac{1}{2}}$$



(c) On the basis of the intuition from part (b), the excess free energy in the case $T < T_c$ would be

$$\Delta G = (\text{Area}) \cdot \left\{ \frac{2}{\xi} N_0^2 + \frac{2}{3} \frac{a}{\xi(T)} \right\}$$

where the second term is the excess free energy of the domain wall as computed in class. Note that the error in the first term due to the approximations made above is larger than the second term.

(d) As described in class, the domain wall solution is actually unstable with respect to a solution in which the magnitude of $\vec{n}$ remains fixed while the direction of the vector rotates into the $\hat{y}$ or $\hat{z}$ direction until it points in the positive $\hat{x}$ direction. This new domain wall solution replaces the third part of the solution in part (b). The total excess free energy is still

$$\Delta G \approx (\text{Area}) \cdot \frac{2}{\xi} N_0^2$$

but the free energy difference of the two solutions is

$$\Delta G_{(c)} - \Delta G_{(d)} = (\text{Area}) \left(\frac{2}{3} \frac{a}{\xi} - \frac{\pi^2 a}{4R} \right)$$

(e) According to this formula, the crossover between the solutions occurs when $R \sim \xi(T)$. Close to $T = T_c$, there is always a region where $\xi(T) \gg R$. We have to go sufficiently far below T_c to exit this region in order to cross over to the second solution. That is $T < T_c - c/aR^2$, where c is of order 1.

In the mechanical analogy, the velocity of the particle at the top of the hill must be very small, so that it can make a sharp turn into the perpendicular direction.