Physics 212 - Quiz #1
Solutions

1.) (a) $$H = -J \sum_{i,\nu}' \vec{\sigma}_i \cdot \vec{\sigma}_{i+\nu}$$

the terms in H that contain the site i are

$$\mathcal{H}_i = -J \sum_{\nu}' \vec{\sigma}_i \cdot (\vec{\sigma}_{i+\nu} + \vec{\sigma}_{i-\nu})$$

$$\langle H_i \rangle = -J \sum_{\nu}' \vec{\sigma}_i \cdot \langle \vec{\sigma}_{i+\nu} + \vec{\sigma}_{i-\nu} \rangle$$

If $\vec{\sigma}_i = 1+ = (1,0)$

$$\langle H_i \rangle = -2dJ\,(P_{1+} - P_{1-})$$

If $\vec{\sigma}_i = 1- = (-1,0)$

$$\langle H_i \rangle = -2dJ\,(-P_{1+} + P_{1-})$$

If $\vec{\sigma}_i = 2+ = (0,1)$

$$\langle H_i \rangle = -2dJ\,(P_{2+} - P_{2-})$$

If $\vec{\sigma}_i = 2- = (0,-1)$

$$\langle H_i \rangle = -2dJ\,(-P_{2+} + P_{2-})$$

The probability of each state, in mean field theory, is proportional to $e^{-\beta \langle H_i \rangle}$

$$p_{1+} = \frac{1}{N} e^{\beta 2dJ (p_{1+} - p_{1-})}$$

$$p_{1-} = \frac{1}{N} e^{-\beta 2dJ (p_{1+} - p_{1-})}$$

$$p_{2+} = \frac{1}{N} e^{\beta 2dJ (p_{2+} - p_{2-})}$$

$$p_{2-} = \frac{1}{N} e^{-\beta 2dJ (p_{2+} - p_{2-})}$$

The sum of these four probabilities should be [illegible], so

$$N = e^{2dJ\beta(p_{1+} - p_{1-})} + e^{-2dJ\beta(p_{1+} - p_{1-})} + e^{2dJ\beta(p_{2+} - p_{2-})} + e^{-2dJ\beta(p_{2+} - p_{2-})}$$

(c) Solve these equations assuming $p_{2+} = p_{2-}$

$$N = e^{2dJ\beta(p_{1+} - p_{1-})} + e^{-2dJ\beta(p_{1+} - p_{1-})} + 2$$

then

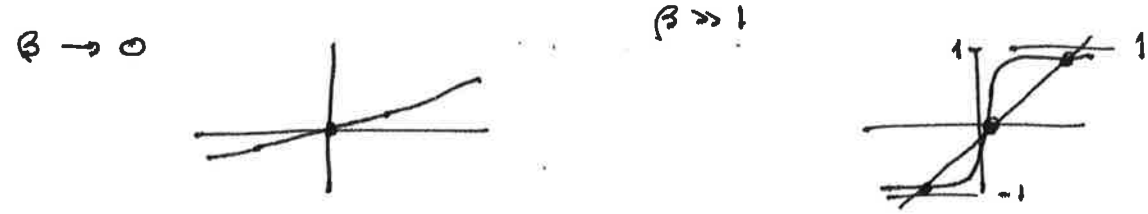
$$p_{1+} - p_{1-} = \frac{e^{2dJ\beta(p_{1+} - p_{1-})} - e^{-2dJ\beta(p_{1+} - p_{1-})}}{e^{2dJ\beta(p_{1+} - p_{1-})} + e^{-2dJ\beta(p_{1+} - p_{1-})} + 2}$$

this is an equation for

$$p_1 = p_{1+} - p_{1-}$$

$$p_1 = \frac{\sinh 2dJ\beta p_1}{\cosh 2dJ\beta p_1 + 1}$$

The right-hand side has the form



The phase transition occurs when the slope of the right-hand side $= 1$ at $p_1 = 0$

$$1 = \frac{d}{dp_1} \frac{2dJ\beta_c p_1}{2 + O(p_1^2)} \Bigg|_{p_1 = 0}$$

$$1 = dJ\beta_c$$

so $\quad \beta_c = \frac{1}{dJ} \quad$ or $\quad T_c = dJ$

d) Now try the ansatz $p_{1+} = p_{2+}$ $\quad p_{1-} = p_{2-}$ Let

$p = p_{1+} - p_{1-} = p_{2+} - p_{2-}$ then

$p_{1+} - p_{1-}$: $\quad p = \frac{1}{N}\left(e^{2dJ\beta p} - e^{-2dJ\beta p}\right)$

$p_{2+} - p_{2-}$: $\quad p = \frac{1}{N}\left(e^{2dJ\beta p} - e^{-2dJ\beta p}\right)$

$$N = \left(e^{2dJ\beta p} + e^{-2dJ\beta p} + e^{2dJ\beta p} + e^{-2dJ\beta p}\right)$$

$$= 2\cosh 2dJ\beta p$$

so

$$p = \frac{\sinh 2d\beta J p}{2\cosh 2d\beta J p}$$

The slope at $p = 0$ is again

$$\frac{2d\beta J}{2} = d\beta J$$

This $= 1$ at $dJ\beta_c = 1$ or $T_c = dJ$

e.) In the first case (part (c)) expanding the self-consistency equation gives

$$p_1 = \frac{2dJ\beta p_1 + \frac{1}{3!}(2dJ\beta p_1)^3 + \cdots}{2 + \frac{1}{2!}(2dJ\beta p_1)^2 + \cdots}$$

$$= dJ\beta p_1 \left(1 + \tfrac{1}{6}(2dJ\beta p_1)^2 - \tfrac{1}{4}(2dJ\beta p_1)^2) + \cdots\right)$$

$$= dJ\beta p_1 \left(1 - \tfrac{1}{12}(2dJ\beta p_1)^2 + \cdots\right)$$

then below T_c $(\beta > \beta_c)$

$$\underbrace{(dJ\beta - 1)}_{\left(\frac{T_c}{T} - 1\right)} p_1 = \frac{4}{12} \underbrace{(dJ\beta)^2}_{\approx 1} p_1^3 + \cdots$$

$$p_1^2 = 3\,\frac{T_c - T}{T}$$

$$p_1 \cong \sqrt{3}\left(\frac{T_c - T}{T_c}\right)^{1/2} \quad \text{and} \quad \vec{M} = N\sqrt{3}\left|\frac{(T_c - T)}{T_c}\right|^{1/4}(1,0)$$

In the second case

$$p = \frac{2dJ\beta p + \frac{1}{3!}(2dJ\beta p)^3 + \cdots}{2(1 + \frac{1}{2!}(2dJ\beta p)^2 + \cdots}$$

$$= dJ\beta p \left(1 + \frac{1}{6}(2dJ\beta p)^2 - \frac{1}{2}(2dJ\beta p)^2 + \cdots\right)$$

$$= dJ\beta p \left(1 + \left(-\frac{4}{3}\right)(dJ\beta p)^2 + \cdots\right.$$

Then, below T_c

$$p \doteq \sqrt{\frac{3}{4}} \left(\frac{T_c - T}{T_c}\right)^{1/2}$$

$$\vec{M} = N\sqrt{\frac{3}{4}} \left(\frac{T_c - T}{T_c}\right)^{1/2} (1, 1)$$

but still $|\vec{M}| = N\sqrt{\frac{3}{2}} \left(\frac{T_c - T}{T_c}\right)^{1/2}$ which is less than in the previous case

f.) Below T_c the value of p_1 is the positive, nontrivial solution to

$$p_1 = \frac{\sinh(2\beta dJ p_1)}{\cosh(2\beta dJ p_1) + 1}$$

p_2 then obeys

$$p_2 = \frac{e^{2d\beta J p_2} - e^{-2d\beta J p_2}}{e^{2d\beta J p_1} + e^{-2d\beta J p_1} + \cosh(2\beta dJ p_1)}$$

$$= \frac{\sinh 2\beta dJ p_2}{\cosh 2\beta dJ p_1 + \cosh 2\beta dJ p_2}$$

For small β, the only solution is at $p_2 = 0$.

For larger β, an instability can occur if the slope of the right-hand side becomes 1. This slope is, for $p_1 > 0$

$$\text{slope} = \frac{2\beta d J}{\cosh 2\beta d J p_1 + 1}$$

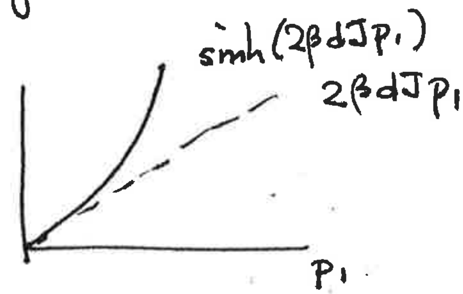
From the p_1 equation

$$\cosh 2\beta d J p_1 + 1 = \frac{\sinh 2\beta d J p_1}{p_1}$$

then

$$\text{slope} = \frac{2\beta d J p_1}{\sinh 2\beta d J p_1}$$

But this is always < 1



so the instability can never occur.