Physics 212 - Quiz #1

Solutions

The partition function of the XY model in 1-d is

$$Z = \sum_{\{\vec{S}_i\}} e^{\beta J \sum_{i=1}^{N} \vec{S}_i \vec{S}_{i+1}}$$

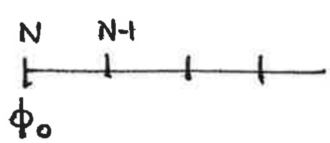
Let $\quad \vec{S}_i = (\cos\phi_i , \sin\phi_i)$

then $\quad \vec{S}_i \cdot \vec{S}_j = \cos(\phi_i - \phi_j)$ and Z can be

written more explicitly as

$$Z = \left(\prod_i \int_{-\pi}^{\pi} \frac{d\phi_i}{2\pi} \right) \prod_i e^{\beta J \cos(\phi_i - \phi_{i-1})}$$

a) Let $Z_N(\phi_o)$ be the partition function for a fixed boundary condition $\phi = \phi_o$



To add one more spin, the operation is

$$Z_{N+1}(\phi) = \int_{-\pi}^{\pi} \frac{d\phi_o}{(2\pi)} e^{\beta J \cos(\phi - \phi_o)} Z_N(\phi_o)$$

so the transfer matrix

$$Z_{N+1} = T Z_n$$

is the linear operator

$$Tf = \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \, e^{\beta J \cos(\phi - \phi')} f(\phi')$$

acting on the Hilbert space of periodic functions of ϕ with period 2π.

b.) To look for eigenvectors of T, try $f_0(\phi) = \text{const} = 1$

$$T f_0 = \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \, e^{\beta J \cos(\phi - \phi')} \cdot 1$$

$$= \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \, e^{\beta J \cos \phi'} \cdot 1 = I_0(\beta J) \cdot 1$$

so this is an eigenfunction with eigenvalue

$$\lambda_0 = I_0(\beta J)$$

c.) Assuming ~~that~~ λ_0 is the largest eigenvalue, for periodic boundary condition

$$Z_N = (\lambda_0)^N = e^{-\beta F}$$

$$F = -\frac{N}{\beta} \log \lambda_0 = -\frac{N}{\beta} \log I_0(\beta J)$$

The asymptotic properties of $I_0(z)$ are

$$I_0(z) = 1 + \tfrac{1}{4}z^2 + \cdots \qquad z \to 0$$

$$I_0(z) = \frac{1}{\sqrt{2\pi z}}\, e^{z} \left(1 + \frac{1}{8z} + \cdots\right) \qquad z \to \infty$$

so as $\beta \to 0$ (high T)

$$F = -\frac{N}{\beta}\,\tfrac{1}{4}(\beta J)^2 = -\frac{N}{4T} J^2$$

as $\beta \to \infty$ low T

$$F = -\frac{N}{\beta}\left[\beta J - \tfrac{1}{2}\log 2\pi\beta J + \cdots\right]$$

So $F \to -NJ$ as $T \to 0$ ✓

d.) This is a translation-invariant problem, so the eigenvectors of T will probably be plane waves. Let's check for waves that are periodic under $\phi \to \phi + 2\pi$

$$f_n(\phi) = e^{in\phi}$$

$$Tf_n = \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi}\, e^{\beta J \cos(\phi-\phi')}\; e^{in\phi'}$$

$-\phi + \phi' = \varphi$

$\phi' = \varphi + \phi$

$$= \int_{-\pi}^{\pi} \frac{d\varphi}{(2\pi)}\, e^{\beta J \varphi}\; e^{in\varphi}\; e^{in\phi}$$

$$= \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi}\, e^{\beta J \varphi} (\cos n\varphi + i \underbrace{\sin n\varphi}_{\to 0 \text{ by symmetry}})\quad e^{in\phi}$$

$$Tf_n = I_n(\beta J)\cdot e^{in\phi} = I_n(\beta J)\cdot f_n$$

so the eigenvectors are

$$f_n \quad e^{in\phi} \qquad n = -\infty \cdots -1, 0, 1, \infty$$

$$\lambda_n \quad I_n(\beta J) \qquad I_{-n}(\beta J) = I_n(\beta J)$$

The asymptotic limits of $I_n(z)$ are

$$I_n(z) = \frac{(\frac{1}{2} z)^n}{n!} + \cdots \qquad z \to 0$$

$$= \frac{e^z}{\sqrt{2\pi z}}\left\{1 - \frac{4n^2-1}{8z} + \cdots\right\}$$

so at large and small z

$$I_0(\beta J) > I_1(\beta J) > \cdots$$

If you look up "Modified Bessel Function of the First Kind" at Wolfram MathWorld (mathworld.wolfram.com) you will see a nice plot showing that this is true for all z.

e.) To compute $\langle \vec{S}_I \vec{S}_J \rangle$, we insert

$$\vec{S}_I\cdot\vec{S}_J = (\cos\phi_I \cos\phi_J + \sin\phi_I \sin\phi_J)$$

into the integral. This is

$$\frac{1}{4}(e^{i\phi_I} + e^{-i\phi_I})(e^{i\phi_J} + e^{-i\phi_J}) + \frac{-1}{4}(e^{i\phi_I} - e^{-i\phi_I})(e^{i\phi_J} - e^{-i\phi_J})$$

$$= \frac{1}{2}(e^{i\Phi_I} e^{-i\Phi_J} + e^{-i\Phi_I} e^{+i\Phi_J})$$

Let's look at the result of inserting $e^{i\phi_I}$:

$$\int \frac{d\phi_I}{2\pi}\; e^{\beta J(\phi_{I+1}-\phi_I)}\; e^{i\phi_I}\; Z_I(\phi_I)$$

If we decompose $Z(\phi_I)$ into eigenvectors, the factor $e^{i\phi_I}$ with convert

$$1 \to e^{i\phi_I} \qquad e^{i\phi_I} \to e^{i2\phi_I} \qquad \text{etc.}$$

$$|0\rangle \to |1\rangle \qquad |1\rangle \to |2\rangle$$

so for the eigenvector $|n\rangle$ multiply by $e^{i\phi_I}$ sends

$$|n\rangle \to |n+1\rangle$$

and multiply by $e^{-i\phi_I}$ sends $|n\rangle \to |n-1\rangle$

Now we can compute $\langle \vec{S}_I \cdot \vec{S}_J \rangle$ for $N \gg I, J$ $\quad J > I$

$$\langle \vec{S}_I \vec{S}_J \rangle = \frac{1}{2}\left\{ \frac{\mathrm{tr}\left[T^{N-J}\, e^{-i\phi_J}\, T^{J-I}\, e^{i\phi_I}\, T^I\right]}{\mathrm{tr}\, T^N} + \left(e^{+i\phi_J} \ldots e^{-i\phi_I}\right)\right\}$$

since $N \to \infty$ $\quad T^N \to |0\rangle \lambda_0^N \langle 0|$. Then the above is

$$= \frac{1}{2}\left\{ \frac{\langle 0|\, e^{-i\phi}\, T^{J-I}\, e^{i\phi}\, |0\rangle\, \lambda_0^{N-(J-I)}}{\lambda_0^N} + \left(e^{i\phi_J} \ldots e^{-i\phi_I}\right)\right\}$$

$$= \frac{1}{2}\left(\frac{\langle 1|\, T^{J-I}\, |1\rangle}{\lambda_0^{J-I}} + \frac{\langle -1|\, T^{J-I}\, |-1\rangle}{\lambda_0^{J-I}} \right) = \left(\frac{\lambda_1}{\lambda_0}\right)^{J-I}$$

since $\lambda_1 = \lambda_{-1}$

Finally $\langle \vec{S}_I \cdot \vec{S}_J \rangle = \left(\frac{\lambda_1}{\lambda_0}\right)^{J-I} = \left[\frac{I_1(\beta J)}{I_0(\beta J)}\right]^{J-I}$

f.) (already seen above)

g.) $\frac{I_1(z)}{I_0(z)} = \frac{1}{2} z \qquad z \to 0$

$$= 1 \cdot \frac{1 - \frac{3}{8}\frac{1}{z}}{1 + \frac{1}{8}\frac{1}{z}} = 1 - \frac{1}{2z} + \cdots$$

$z \to \infty$

then for high T

$$\langle \vec{S}_I \cdot \vec{S}_J \rangle = \left(\frac{\beta J}{2}\right)^{J-I} = \exp\left[|J-I| \log \frac{\beta J}{2} \right]$$

$$= \exp\left[|J-I| / \xi(T) \right]$$

$$\xi(T) = \frac{1}{\log J/2T} \to 0 \quad \text{for } T \to \infty$$

for low T

$$\langle \vec{S}_I \cdot \vec{S}_I \rangle = \left(1 - \frac{1}{2z}\right)^{J-I}$$

$$= \exp\left[-|J-I| \frac{1}{2\beta J} \right]$$

so $\xi = \frac{2J}{T} \to \infty$ for $T \to 0$

h.)

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In[1]:= Quit[];

In[2]:= cratio[beta_] := BesselI[1, beta] / BesselI[0, beta];

In[3]:= xi[beta_] := -1 / Log[cratio[beta]];

In[10]:= Plot[xi[1 / T], {T, 0.1, 15}, PlotRange → {{0, 15}, {0, 2}}]
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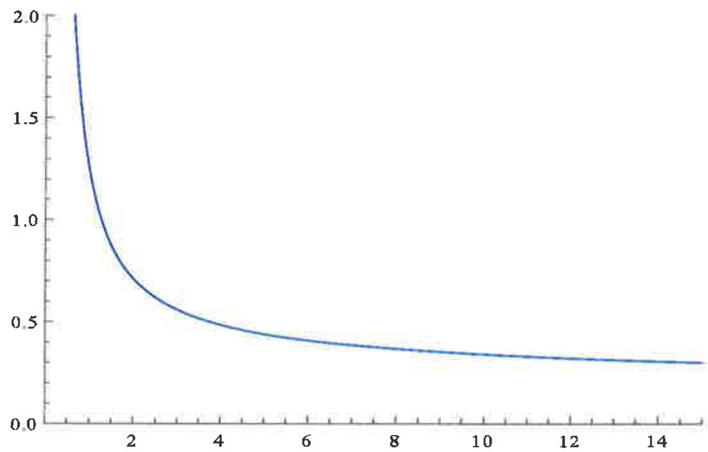
Out[10]=



check asymptotes:

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In[11]:= Plot[{xi[1 / T], 2 / T, 1 / Log[2 T]}, {T, 0.1, 15}, PlotRange → {{0, 15}, {0, 6}}]
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Out[11]=