Physics 212 – Statistical Mechanics

Quiz #1 - Solution

1. (a) From the partition function

$$Z = \sum_S e^{-\beta H[S]} = \sum_S e^{\beta J \sum S_i S_{i+\nu} + \beta H \mu \sum_i S_i}$$

we have

$$M = \mu \sum_i \langle S_i \rangle = \frac{\sum_S e^{-\beta H} \mu \sum_i S_i}{\sum_S e^{-\beta H}}$$

Then

$$\chi = \frac{\sum_S e^{-\beta H} \mu \sum_i S_i \cdot \beta\mu \sum_j S_j}{\sum_S e^{-\beta H}} - \frac{\left(\sum_S e^{-\beta H} \mu \sum_i S_i\right)\left(\sum_S e^{-\beta H} \beta\mu \sum_j S_j\right)}{\left(\sum_S e^{-\beta H}\right)^2}$$

so that

$$\chi = \beta\mu^2 \sum_{ij} \left(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle\right)$$

In the high-temperature phase, $\langle S_j \rangle = 0$. Also, $\langle S_i S_j \rangle$ is translation invariant,

$$\langle S_i S_j \rangle = \langle S_{i-j} S_0 \rangle$$

Then

$$\chi = \beta\mu^2 N \sum_j \langle s_j s_0 \rangle$$

(b) As shown in class, the partition function at $H = 0$ is given by

$$Z = \sum_s \prod_{\text{bonds}} \cosh\beta J \,(1 + s_i s_{i+r} \tanh\beta J)$$
$$= 2^N (\cosh\beta J)^{2N} \sum_{n=0}^{\infty} b_n z^n$$

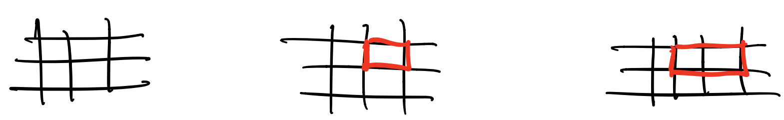
where b_n is the number of colorings of the lattice with n colored bonds such that there is an even number of colored bonds at each vertex. Also

$$\sum_s e^{-\beta H} s_j s_0 = 2^N (\cosh\beta J)^{2N} \sum_{n=0}^{\infty} c_n^j z^n$$

where c_n^j is the the number of colorings of the lattice with n colored bonds such that there is an even number of colored bonds at each vertex – except that, at the vertices 0 and j, there are an odd number of colored bonds. We wish to compute

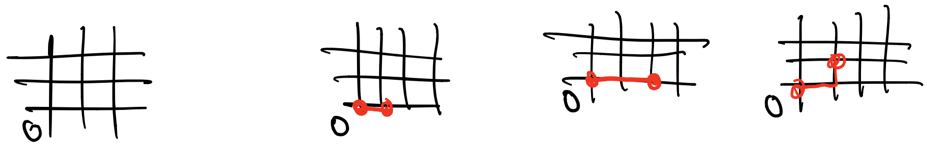
$$\chi = \beta\mu^2 \frac{\sum_{n=0}^{\infty} \sum_j c_n^j z^n}{\sum_{n=0}^{\infty} b_n z^n}$$

The value of the denominator in this expression is



$$1 + N z^4 + O(z^6)$$

The first few terms in the numerator are

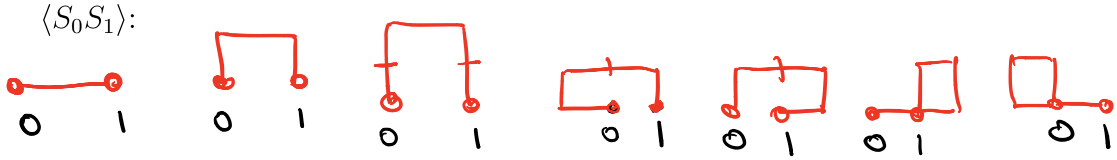


$$1 + 4\cdot z + 4\cdot 3\, z^2 + \cdots$$

There are many was to count the colorings in higher orders. Brute force is not so painful. Here are the diagrams that contribute to $\langle S_0 S_0\rangle$ through order z^5:

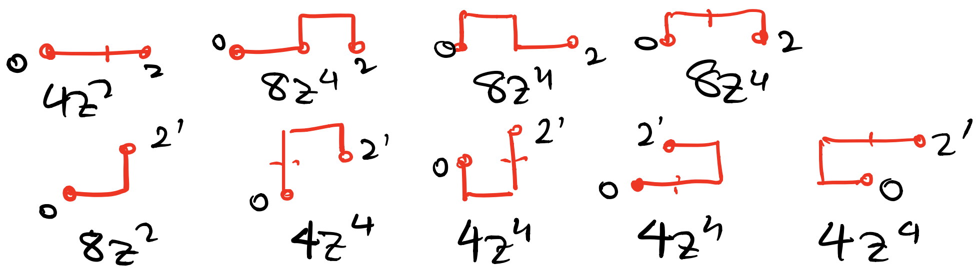
$$1 + 4z^4$$

plus reflections; the total number of diagrams of each type is indicated by the number written below. Here are the diagrams that contribute to $\langle S_0 S_1\rangle$:



$$4z \qquad 8z^3 \qquad 8z^5 \qquad 8z^5 \qquad 8z^5 \qquad 8z^5 \qquad 8z^5$$

Here are the diagrams that contribute to $\langle S_0 S_2\rangle$ and $\langle S_0 S_{2'}\rangle$:



We see that the diagrams resemble random walks starting at 0. Also the diagrams with j separated from 0 by n or more steps contain no closed loops up to order z^{n+4}.

Given this understanding, here is an easier way to sum the diagrams. Let's refer to walks starting from 0 and not backtracking as "snakes". The number of snakes with n steps is

$$1,\ 4,\ 4\cdot 3,\ 4\cdot 3^2,\ 4\cdot 3^3,\ 4\cdot 3^4,\ \cdots$$

$$= 1,\ 4,\ 12,\ 36,\ 108,\ 324,\ \cdots$$

The snakes include all allowed diagrams, but some diagrams appear two or more times. This can only happen if the diagrams contain closed loops. The first example appears in order z^4. The loop diagram above appears in the snakes

and so is overcounted by 4. At order z^4 there are also diagrams contributing to $\langle S_0 S_0 \rangle$ in which the closed loop does not touch the point 0.

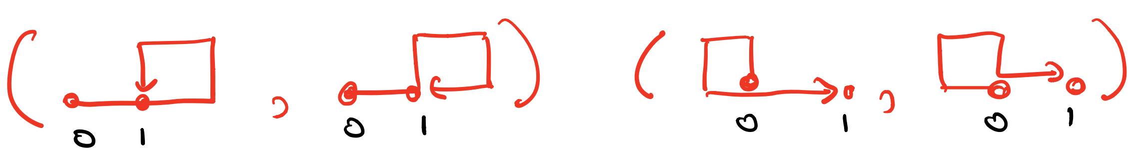
$$(N-4)z^4$$

In all, the coefficient of z^4 is

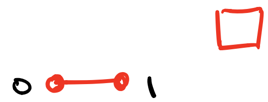
$$108 - 4 + N - 4 = 100 + N$$

We can apply the same logic to $n = 5$. The snakes

do not correspond to any allowed coloring, an overcount of 8. The pairs of snakes



yield the same coloring. This is an overcount of $2 \cdot 8 = 16$. There are also $4(N-6)$ disconnected diagrams with a square that does not touch the line emerging from 0



$$4(N-6)$$

In all, the value of $\sum_j c_5^j$ is

$$324 - 8 - 16 + 4N - 24 = 276 + 4N$$

In all

$$\chi/\beta\mu^2 = \frac{1 + 4z + 12z^2 + 36z^3 + (100+N)z^4 + (276+4N)z^5}{1 + Nz^4 + \cdots}$$

or

$$\chi/\beta\mu^2 = 1 + 4z + 12z^2 + 36z^3 + 100z^4 + 276z^5 + \cdots$$

The N's cancel from the coefficients, as they must.

(c) The Taylor expansion of $A/(z_* - z)^\gamma$ is

$$\frac{A}{(z_* - z)^\gamma} = \frac{A}{z_*^\gamma} \frac{1}{(1 - z/z_*)^\gamma} = \frac{A}{z_*^\gamma}\left(1 + \gamma \frac{z}{z_*} + \frac{\gamma(\gamma+1)}{2!}\left(\frac{z}{z_*}\right)^2 + \cdots\right.$$

Then the ratio of successive coefficients is

$$\frac{a_n}{a_{n-1}} = \frac{\gamma + n - 1}{n} \frac{1}{z_*}$$

Asymptotically

$$\frac{a_n}{a_{n-1}} \longrightarrow \frac{1}{z_*}$$

(d) Trying this on the series coefficients in (b), the ratios of successive series coefficients are

$$\frac{a_n}{a_{n-1}} = 4,\ 3,\ 3,\ 2.78,\ 2.76,\ \cdots$$

This points to

$$\frac{1}{z_*} \approx 2.8$$

or

$$\tanh \beta J \approx 0.39 \qquad \frac{T_c}{J} \cong 2.7$$

In principle, if we had enough terms in the series, we could use

$$\frac{a_{n-1}}{a_{n-2}} - \frac{a_n}{a_{n-1}} = \frac{\gamma - 1}{n(n-1)}$$

to estimate γ. The actual value of γ for the 2-dimensional Ising model is $7/4 = 1.75$. This implies that the ratios of successive terms should increase asymptotically. This short series does not yet show that behavior.