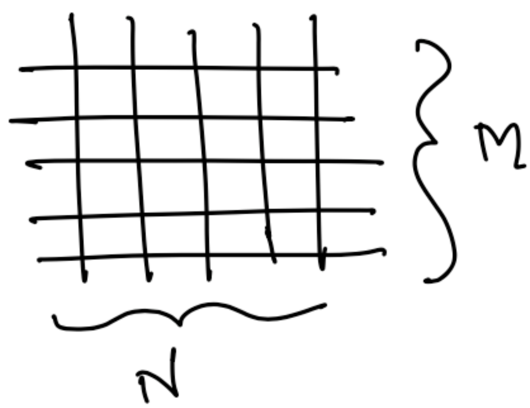
Oct. 12

Physics 212 – Statistical Mechanics

Solution of the 2-Dimensional Ising Model

In the previous lecture, we saw that mean-field theory is incorrect for the 1-dimensional Ising model. What is the situation in 2 dimensions? In this case also, the partition function can be calculated exactly. The solution is not as easy as in 1 dimension; in fact, the first solution, by Lars Onsager in 1942, was a tour de force of mathematical physics. Over the years, this solution has been simplified, but still it is not easy. Nevertheless, the solution gives insights that are fascinating and important. In this lecture, I will present the main features of that solution following the method of Schultz, Mattis, and Lieb (Rev. Mod. Phys. 26, 856 (1964)).

I will consider a 2-dimensional Ising model for $H = 0$ on a square lattice of dimensions $N \times M$, with periodic boundary conditions,



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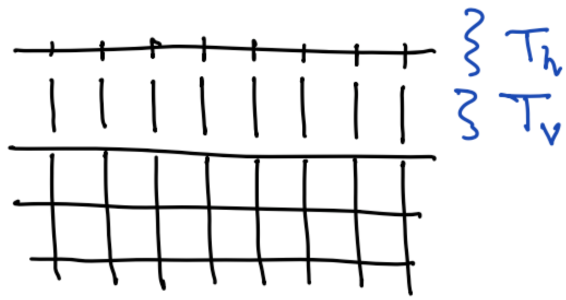
To solve the problem, I will construct the transfer matrix T for this system such that

$$Z = \mathrm{tr}[T^M] \ , \tag{2}$$

and I will explain how to diagonalize this matrix. In the previous lecture, T was a 2×2 matrix that was easily diagonalized. In this case, T acts on all spins in a row of length N, so it is $2^N \times 2^N$.

The transfer matrix is defined as the matrix that adds one layer of spins. We can imagine adding this layer in two stages, first, add the vertical bonds, then, add the horizonal bonds.



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This defines two matrices T_v and T_h, acting on the state of spin configurations on a line

$$|\Psi\rangle = |S_1 S_2 S_3 \cdots S_N\rangle \tag{4}$$

I assume periodic boundary conditions, so that $S_{N+1} = S_1$. Let σ_j^a be the action of one of the Pauli sigma matrices on the jth spin. We add the horizontal bonds with a matrix T_h whose matrix elements are

$$\langle S_1' S_2' \cdots | \, T_h \, | S_1 S_2 \cdots \rangle = \delta_{S_1', S_1} \delta_{S_2', S_2} \cdots \exp\left[\beta J \sum_j S_j S_{j+1}\right] \tag{5}$$

Then, T_h can be represented by the operator

$$T_h = \exp\left[\beta J \sum_j \sigma_j^z \sigma_{j+1}^z\right] . \tag{6}$$

We add the vertical bonds with a matrix T_v whose matrix elements are

$$\langle S_1' S_2' \cdots | \, T_v \, | S_1 S_2 \cdots \rangle = \exp\left[\sum_j \beta J S_j' S_j\right] \tag{7}$$

It is worth a little effort to simplify this. The right-hand side is a product of factors

$$e^{\beta J S' S} = \begin{cases} e^{\beta J} & S' = S \\ e^{-\beta J} & S' = -S \end{cases} \tag{8}$$

corresponding to a matrix action on the a 2-dimensional Hilbert space for each spin

$$\begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} = e^{\beta J} \cdot \begin{pmatrix} 1 & e^{-2\beta J} \\ e^{-2\beta J} & 1 \end{pmatrix} \tag{9}$$

Apart from the prefactor, this matrix depends on $e^{-2\beta J}$. Recall from our discussion of the low-temperature series of the Ising model that it was convenient to define

$$e^{-2\beta J} = w \tag{10}$$

and that this variable w was related to the temperature $\overline{T}$ of the *Kramers-Wannier dual* Ising model by $w = \bar{z}$, that is

$$w = \tanh \overline{\beta} J \tag{11}$$

Then we can write (8) in the form

$$\frac{e^{\beta J}}{\cosh \overline{\beta} J}(\cosh \overline{\beta} J + \sigma^x \sinh \overline{\beta} J) = \frac{e^{\beta J}}{\cosh \overline{\beta} J} \cdot \exp[\overline{\beta} J \sigma^x] \tag{12}$$

From here on, I am going to drop the prefactors $e^{\beta J}/\cosh \overline{\beta} J$. These contribute a factor in the free energy that is analytic in β and so does not affect the physics of the critical point. Finally, then, we have

$$T_v = \exp\Big[\overline{\beta} J \sum_j \sigma^x_j\Big] \ . \tag{13}$$

Let $K = \beta J$ and $\overline{K} = \overline{\beta} J$. I would like to prove a simple identity that we will use later. Since $\tanh \overline{K} = e^{-2K}$,

$$\begin{aligned} \sinh 2K &= \frac{1}{2}\Big(\frac{1}{\tanh \overline{K}} - \tanh \overline{K}\Big) \\ &= \Big(\frac{\cosh^2 \overline{K} - \sinh^2 \overline{K}}{2 \cosh \overline{K} \sinh \overline{K}}\Big) = \frac{1}{\sinh 2\overline{K}} \end{aligned} \tag{14}$$

Then

$$\sinh 2K \cdot \sinh 2\overline{K} = 1 \tag{15}$$

At the critical point, $K = \overline{K}$ and so $\sinh 2K = \sinh 2\overline{K} = 1$. You can check that this agrees with the result for the critical temperature derived in last week's lecture. Note that $\overline{K} > K$ in the high-temperature region.

Now let me use a small trick: exchange $\sigma^x \to -\sigma^z$, $\sigma^z \to \sigma^x$. This rotation in the spin space has no effect on the value of the trace. With this rotation,

$$T_h = \exp\Big[K \sum_j \sigma^x_j \sigma^x_{j+1}\Big] \ , \quad T_v = \exp\Big[-\overline{K} \sum_j \sigma^z_j\Big] \ , \tag{16}$$

Then we can construct a Hermitian T by

$$T = T_h^{1/2} T_v T_h^{1/2} \tag{17}$$

or vice versa. In either case, the partition function is given by

$$Z = \mathrm{tr}[T^M] \ . \tag{18}$$

The matrices T_h and T_v do not commute, so we cannot directly combine the exponentials. This leads to some complication in the diagonalization of the transfer matrix. I make this lecture a little simpler, I am going to ignore this difficulty and claim that we can get the essential results from the similar matrix

$$T' = \exp[-\mathcal{H}] \ , \tag{19}$$

ith

$$\mathcal{H} = -K \sum_j \sigma_j^x \sigma_{j+1}^x + \overline{K} \sum_j \sigma_j^z \tag{20}$$

Then the largest eigenvalue of the transfer matrix comes from the ground state of the Hamiltonian $\mathcal{H}$. It can be shown that the full transfer matrix (17) can be diagonalized by the same methods that I will use below, giving very similar results. See the paper of Schultz, Mattis, and Lieb for the details.

Notice that the trace of each term in $\mathcal{H}$ is equal to zero, so

$$\mathrm{tr}[\mathcal{H}] = 0 \ . \tag{21}$$

This will be a useful touchstone in our discussion later.

We still have the problem of simplifying the spin chain Hamiltonian (20). This is still not so easy, since this Hamiltonian represents a quantum system with N interacting degrees of freedom. I will show that it can be solved by a very interesting trick.

The Hamiltonian (20) is translation-invariant, so it is useful to go to Fourier space. For functions on a lattice of N sites, any function f_j can be represented by a Fourier series

$$f_j = \frac{1}{\sqrt{N}} \sum_q e^{iqj} \tilde{f}_q \ , \qquad f_q = \frac{1}{\sqrt{N}} \sum_j e^{-iqj} \tilde{f}_j \tag{22}$$

For concreteness, I will assume that N is even. Then if f_j has periodic boundary conditions, $f_{N+1} = f_1$, then q runs over the values

$$q = 0, \pm\frac{2\pi}{N}, \pm\frac{4\pi}{N}, \cdots, \pm\frac{(N-2)\pi}{N}, \pi \tag{23}$$

If f_j has antiperiodic boundary conditions $f_{M+1} = -f_1$, then q takes the values

$$q = \pm\frac{\pi}{N}, \pm\frac{3\pi}{N}, \cdots, \pm\frac{(N-1)\pi}{N} \tag{24}$$

In either case, there are N values f_j and N values f_q.

Unfortuntely, it is awkward to Fourier transform matrix functions like σ_i^x, σ_i^z, since these have fixed normalizations at each site. We can avoid this problem by a change of variables. First, decompose

$$\sigma^x = \sigma^+ + \sigma^- \tag{25}$$

with

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{26}$$

Then define

$$\Sigma_j = \prod_{k=1}^{(j-1)} (-\sigma_k^z) \tag{27}$$

Then $\Sigma_j = (-1)^n$, where n is the number of up spins on the sites to the left of the site j. Finally, let

$$\psi_j = \Sigma_j \sigma_j^- \qquad \psi^\dagger = \Sigma_j \sigma_j^+ \tag{28}$$

This is called the *Jordan-Wigner transformation.*

The operator Σ_k is a disorder operator based at k.

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In some sense, the new object we have constructed is the product of an order operator and a disorder operator

$$\psi_j \sim \Sigma_j S_j \tag{30}$$

I will make this relation more precise at a later point in the course.

Spin operators have nontrivial commutation relations at the same site

$$\{\sigma_j^+, \sigma_j^-\} = 1 \qquad \{\sigma_j^-, \sigma_j^-\} = 0 \tag{31}$$

but they commute on different sites. Thus, the spin is an interacting boson. The operator ψ_j has a different property. On the same site, it is still true that

$$\{\psi_j, \psi_j^\dagger\} = 1 \qquad \{\psi_j, \psi_j\} = 0 \ , \tag{32}$$

but on different sites, the ψ_j *anticommute.* Since σ_k^- lowers one spin, If $k < j$,

$$\Sigma_j \sigma_k^- = -\sigma_k^- \Sigma_j \ \text{ while } \ \Sigma_k \sigma_j^- = +\sigma_j^- \Sigma_k \tag{33}$$

Then

$$\begin{aligned} \{\psi_j, \psi_k\} &= \Sigma_j \sigma_j^- \Sigma_k \sigma_k^- + \Sigma_k \sigma_k^- \Sigma_j \sigma_j^- \\ &= (\Sigma_j \Sigma_k - \Sigma_k \Sigma_j) \sigma_j^- \sigma_k^- = 0 \ . \end{aligned} \tag{34}$$

That is, ψ_j, $\psi_j^\dagger$ obey the commutation relations

$$\{\psi_j, \psi_k^\dagger\} = \delta(j-k) \ , \qquad \{\psi_j, \psi_k\} = 0 \tag{35}$$

These are the commutation relations of creation and annihilation operators that we discussed in the first lecture, except that, here, they are anti-commutation relations. Thus, they relate to a different kind of oscillator with the operator algebra

$$\{a, a^\dagger\} = 1 \ , \qquad \{a, a\} = 0 \tag{36}$$

This algebra describes a Hilbert space with only two states,

$$|0\rangle \ , \quad |1\rangle = a^\dagger |0\rangle \tag{37}$$

You can see from the anticommutation relation that

$$a |1\rangle = |0\rangle \ , \tag{38}$$

and that the state

$$(a^\dagger)^2 |0\rangle \tag{39}$$

does not exist, because $(a^\dagger)^2 = 0$. This oscillator has a state that can only be occupied once. Thus, it describes a fermion. More generally, (35) gives the commutation relations for *fermionic* creation and annihilation operators. If we turn the lattice into a continuum and then go to Fourier space, we have operators

$$a_p = \int dx \ e^{-ipx} a(x) \ . \tag{40}$$

and these satisfy the relations

$$\{a_p, a_k^\dagger\} = (2\pi)\delta(p - k) \qquad \{a_p, a_k\} = 0 \ . \tag{41}$$

The Hilbert space spanned by the operators $a_k^\dagger$ acting on $|0\rangle$ is the Fock space of fermions. We can also create a fermionic quantum field, exactly analogous to the bosonic quantum field that we wrote in the first lecture.

In 3 dimensions, you cannot turn a boson into a fermion. There is a results which holds generally in our observations — and, actually, can be proved rigorously assuming local quantum field theory — called the "Spin-Statistics Theorem": Any particle with integer spin must be a boson, and any particle with half-integer spin must be a fermion. However, spin refers to the quantum number under the rotation group in 3 dimensions. In 1 dimension, there is no rotation group, and so there is no barrier that keeps us from changing bosons into fermions or vice versa. Actually, it is easy to do this, as Jordan and Wigner showed.

In our 1-dimensional spin chain, the operator that counts the number of fermions at the site j is

$$\psi_j^\dagger \psi_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j = \frac{1}{2}(\sigma_j^z + 1) \ , \tag{42}$$

We can then identify

$$\sigma_j^z = 2\psi_j^\dagger \psi_j - 1 \tag{43}$$

Similarly, the operator $\sigma_j^x \sigma_{j+1}^x$ in $\mathcal{H}$ can be rewritten in pieces as

$$\begin{aligned} \sigma_j^+ \sigma_{j+1}^+ &= \psi_j^\dagger \psi_{j+1}^\dagger \qquad & \sigma_j^+ \sigma_{j+1}^- &= \psi_j^\dagger \psi_{j+1} \\ \sigma_j^- \sigma_{j+1}^+ &= -\psi_j \psi_{j+1}^\dagger \qquad & \sigma_j^- \sigma_{j+1}^- &= -\psi_j \psi_{j+1} \end{aligned} \tag{44}$$

To go from the left-hand side to the right-hand side in (44), we add in the operators Σ_j and Σ_{j+1}. When we put the factor $(-\sigma_j^z)$ in Σ_{j+1} in front of σ_j^+, it gives a factor $(+1)$, because σ^+ is nonzero only on a state with spin down. Similarly, when we put the factor $(-\sigma_j^z)$ in Σ_{j+1} in front of σ_j^-, it gives a factor (-1), because σ^- is nonzero only on a state with spin up.

This change of variables brings $\mathcal{H}$ into the form

$$\mathcal{H} = -K \sum_j (\psi_j^\dagger - \psi_j)(\psi_{j+1}^\dagger + \psi_{j+1}) + \overline{K} \sum_j (2\psi_j^\dagger \psi_j - 1) \ . \tag{45}$$

Notice that, still, $\text{tr}[\mathcal{H}] = 0$.

Now we are free to Fourier transform the various field operators. I would like to slightly modify the above definitions to remove some minus signs in $\mathcal{H}$. Then, define

$$\psi_j = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_q e^{iqj} \psi_q \qquad \psi_j^\dagger = \frac{e^{+i\pi/4}}{\sqrt{N}} \sum_q e^{-iqj} \psi_q^\dagger \tag{46}$$

Inserting these expressions, the second term of $\mathcal{H}$ becomes simply

$$\overline{K} \ \sum_q (2\psi_q^\dagger \psi_q - 1) \ . \tag{47}$$

The first term becomes

$$\begin{aligned} -K \ &\sum_q (\psi_q^\dagger - i\psi_{-q})(-ie^{iq}\psi_{-q}^\dagger + e^{iq}\psi_q) \\ &= -K \sum_{q\geq 0} \Big(2\cos q(\psi_q^\dagger \psi_q + \psi_{-q}^\dagger \psi_{-q}) + 2\sin q(\psi_{-q}\psi_q + \psi_q^\dagger \psi_{-q}) \Big) \ . \end{aligned} \tag{48}$$

To get all of the signs right in this expression, you will need to use the fact that the operators ψ_q and $\psi_q^\dagger$ anticommute, so that, for example, $\psi_{-q}\psi_{-q}^\dagger = -\psi_{-q}^\dagger \psi_{-q}$, plus a constant.

Amazingly, these changes of variables convert $\mathcal{H}$ into a quadratic form in fermion operators. That is, the problem of nonlinearly interacting spins (bosons) has been transformed into a problem of linear (free!) fermions. The full Hamiltonian is now

$$\mathcal{H} = -K \sum_{q\geq 0} \Big(2\cos q(\psi_q^\dagger \psi_q + \psi_{-q}^\dagger \psi_{-q}) + 2\sin q(\psi_{-q}\psi_q + \psi_q^\dagger \psi_{-q}) \Big) + \overline{K} \sum_{q\geq 0} 2(\psi_q^\dagger \psi_q + \psi_{-q}^\dagger \psi_{-q}) \tag{49}$$

plus a constant to keep $\mathrm{tr}[\mathcal{H}] = 0$.

The periodicity properties of ψ_j will be important. Note that

$$\psi_{N+1} = \prod_{i=1}^{N}(-\sigma_i^z) \cdot \psi_1 \tag{50}$$

The prefactor is $(-1)^F$, where F is now the total number of fermions in the state. So, if there are an even number of fermions in the state, ψ_k obeys periodic boundary conditions and, if there are an odd number of fermions in the state, ψ_k obeys antiperiodic boundary conditions.

In the even case, we have the isolated momentum states $q = 0$ and π. These contribute to $\mathcal{H}$ the terms

$$\mathcal{H}_{0,\pi} = (-2K + 2\overline{K})(\psi_0^\dagger\psi_0 - \frac{1}{2}) + (+2K + 2\overline{K})(\psi_\pi^\dagger\psi_\pi - \frac{1}{2}) \tag{51}$$

where I have added constants so that $\mathrm{tr}[\mathcal{H}_{0,\pi}] = 0$. Note that, for $\overline{K} > K$, high temperature, the ground state has the mode $q = 0$ unoccupied while for $K > \overline{K}$, low temperature, the ground state has the mode $q = 0$ occupied.

For the other q modes, we have the contribution to $\mathcal{H}$

$$\mathcal{H}_q = 2(\overline{K} - K\cos q)(\psi_q^\dagger\psi_q + \psi_{-q}^\dagger\psi_{-q}) + 2K\sin q(\psi_{-q}\psi_q + \psi_q^\dagger\psi_{-q}) \ . \tag{52}$$

plus a constant to insure that $\mathrm{tr}[\mathcal{H}_q] = 0$. Let's diagonalize this in the case $\overline{K} > K$. Let

$$\begin{aligned} \chi_q &= \alpha\psi_q + \beta\psi_{-q}^\dagger & \qquad \chi_{-q} &= \alpha\psi_{-q} - \beta\psi_q^\dagger \\ \chi_q^\dagger &= \alpha\psi_q^\dagger + \beta\psi_{-q} & \qquad \chi_{-q} &= \alpha\psi_{-q}^\dagger - \beta\psi_q^\dagger \end{aligned} \tag{53}$$

with

$$\alpha = \cos\theta_q/2 \ , \qquad \beta = \sin\theta_q/2 \tag{54}$$

such that

$$\cos\theta_q = 2(\frac{\overline{K} - K\cos q}{\epsilon_q}) \ , \qquad \sin\theta_q = -2(\frac{K\sin q}{\epsilon_q}) \ , \tag{55}$$

and

$$\epsilon_q^2 = 4[(\overline{K} - K\cos q)^2 + (K\sin q)^2] \tag{56}$$

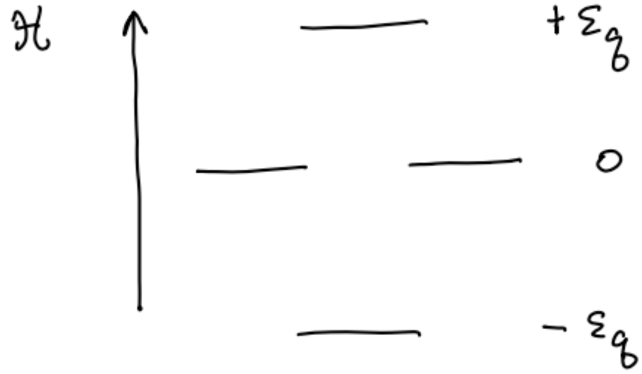
so that

$$\epsilon_q = 2\Big[\overline{K}^2 + K^2 - 2K\overline{K}\cos q\Big]^{1/2} \ . \tag{57}$$

Then

$$\mathcal{H}_q = \epsilon_q \ (\chi_q^\dagger\chi_q + \chi_{-q}^\dagger\chi_{-q} - 1) \ . \tag{58}$$

I have supplied a constant so that $\mathrm{tr}[\mathcal{H}_q] = 0$, insuring that $\mathrm{tr}[\mathcal{H}] = 0$. We find the spectrum



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The ground state is an empty state at $\mathcal{H} = -\epsilon_q$, and the higher energy levels are obtained by filling one or both of the fermion states. For $K > \overline{K}$, there is a similar structure.

Note that, for small q, the expression for ϵ_q takes the form

$$\epsilon_q = A[C^2 + q^2]^{1/2} \tag{60}$$

Thus, the energy spectrum of the model near $q = 0$ is that of a relativistic free fermion, with a mass C that goes to zero just at the critical temperature where $\overline{K} = K$.

Now we are prepared to work out the ground state energy of $\mathcal{H}$. Each q sector is independent, and in each sector the lowest state is one at an energy of $-\epsilon_q$. Thus, the ground state energy of $\mathcal{H}$ is

$$E_0 = -\sum_{q>0} \epsilon_q \tag{61}$$

up to some small (but crucial) effects that we are about to discuss.

The formula (61) gives the ground state energy of $\mathcal{H}$ in general terms, but now we need to find in all detail the explicit state that is lowest in energy. This is straightforward when $\overline{K} > K$. In each sector, including $q = 0$, the fermion occupation number is zero. Then the total number of fermions is even, so it is consistent to have the $q = 0$ mode, unoccupied. This is the unique ground state of $\mathcal{H}$.

However, when $K > \overline{K}$, the answer to this question is more complicated. If the number of fermions is even, the $q = 0$ mode exists, but then in the ground state it is occupied, and so there must be another occupied state among the nonzero values of q. Thus, the energy of that the ground state gets an additional contribution of $+\epsilon_q$ for $q \approx 0$. Alternatively, we can consider a solution in which the total number of fermions is odd. In this case, in which the one of the nonzero q states will be occupied. Since the total number of fermions is odd, the mode at $q = 0$ does not exist. The energy of this state, above the unoccupied state, is also $+\epsilon_q$ for $q \approx 0$. These two states have the same energy in the thermodynamic limit $N \to 0$ in which the q modes form a continuum. Thus, below T_c, there are two distinct spaces of eigenvectors with the same leading eigenvalue of the transfer matrix. This is what is required to have a magnetized phase.

There are a few more things that one can learn about the 2-dimensional Ising model from this picture:

1. It is important to note that the whole structure of the ground state of $\mathcal{H}$ and thus the leading eigenvalue of the transfer matrix changes at $K = \overline{K}$, that is, at $T = T_c$. This change in behavior leads to singularities that must show up in thermodynamic functions.

2. For $T > T_c$, there is a unique ground state of T. Then, by the arguments of the previous lecture, the spin-spin correlation function $\langle S_I S_J \rangle$ must go to zero exponentially for large separation. However, for $T < T_c$, this state is no longer the ground state, there are two different sectors with degenerate ground states, and we are allowed to have spin correlations of infinite range

3. From (61), we see that the free energy of our system is

$$F = -\frac{1}{\beta}\log Z = -\frac{M}{\beta}(-\sum_{q>0}\epsilon_q) \ . \tag{62}$$

Taking into account that C above vanishes at $T = T_c$, this has the structure at small q

$$F = \frac{M}{\beta}A \sum_q \, [c(T-T_c)^2 + q^2]^{1/2} \ . \tag{63}$$

Passing from the sum over q to an integral, we have

$$F \approx \frac{AMN}{\beta}\int_0^\pi \frac{dq}{2\pi} \, [c(T-T_c)^2 + q^2]^{1/2} \ . \tag{64}$$

Then, the thermodynamic energy contains a term

$$E = \frac{\partial}{del T} \, F = \frac{AMN}{\beta}\int \frac{dq}{2\pi} \, \frac{c(T-T_c)}{[c(T-T_c)^2 + q^2]^{1/2}} \ . \tag{65}$$

and the specific heat has a term

$$C = \frac{\partial E}{\partial T} = \frac{AMN}{\beta}\int \frac{dq}{2\pi} \, \frac{c}{[c(T-T_c)^2 + q^2]^{1/2}} \ . \tag{66}$$

For $T - T_c$ small, this integral has a logarithmic singularly as $q \to 0$. Then

$$C \sim \log \frac{1}{|T-T_c|} \tag{67}$$

This is a different behavior from what we found in mean field theory.

4. It can be shown that the correlation function of massless relativistic fermions in 1 dimension behaves as

$$\left\langle \psi(x)\psi^{\dagger}(y) \right\rangle \sim \frac{1}{|x-y|} \tag{68}$$

This is an interesting piece of data that will contribute to the general picture we will derive later. It is also possible to compute the spin-spin correlation function at T_c. This can be shown to behave as

$$\langle S(x)S(y) \rangle \sim \frac{1}{|x-y|^{1/4}} \ . \tag{69}$$

The exponent here is very peculiar. I will derive the power laws (68) and (69) in a very surprising way in the last lecture of the course.

5. It is also possible to use this formalism to compute the spontaneous magnetization of the Ising model for $T < T_c$ The analysis requires some more sophisticated mathematical tricks; see Schultz, Mattis, and Lieb for details. The result for the magnetization at zero field was announced by Onsager in 1944, but he did not publish the derivation. The first published derivation of the spin-spin correlation function was given in 1949 by a then-unknown Chinese postdoc working in Enrico Fermi's group in Chicago – C. N. Yang. The result was found to be

$$\langle S_i \rangle = \left[1 - \frac{1}{\sinh^2 2K}\right]^{1/8} \tag{70}$$

that is

$$M \sim (T_c - T)^{1/8} \tag{71}$$

So, there is a nonzero spontaneous magnetization for $T < T_c$, but the exponent with which this magnetization goes to zero as $T \to T_c$ is completely different from the one predicted by mean field theory.

In all, the exact solution to the 2-dimensional Ising model supports the qualitative picture of the Ising model phase diagram given by mean field theory. However, the detailed quantitative description of the thermodynamic functions near T_c is different. This gives us a certain amount of food for thought for the questions that we will discuss in the rest of the course.