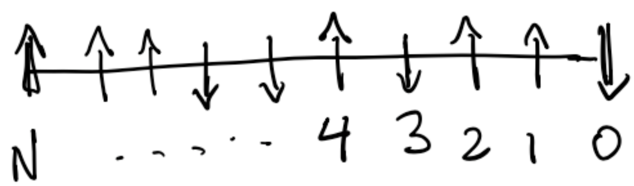
Oct. 10

Physics 212 – Statistical Mechanics
Solution of the 1-Dimensional Ising Model

It happens that, for the the cases of the 1- and 2-dimensional Ising models, the partition function can be computed exactly. The 1-dimensional case was of course solved by Ising. In this lecture and the next, I will discuss these solutions. The results and the methods contain some surprising insights.

Let's begin with the 1-dimensional Ising model,



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To start, I will set up the computation for a finite number of spins N with the boundary conditions that the spins at the ends are fixed. An example would be to take $S_0 = S_N = +1$ and sum over all other spins. The ground state of this system, realized at $T = 0$, is the state will all spins up. According to our analyses last week, there ought to be a magnetized state that persists at finite T. Is it correct?

The partition function that we would like to compute is

$$Z_N(S_n, S_0) = \sum_{s_i=\pm 1} e^{\beta J \sum_{i=0}^{N-1} S_i S_{i+1} + \beta H \sum_{i=1}^{N-1} S_i} \tag{2}$$

Let's consider the smallest values of N and work our way up. For $N = 1$,

$$Z_1 = e^{\beta J S_0 S_1} \tag{3}$$

For $N = 2$,

$$\begin{aligned} Z_2(S_0, S_2) &= \sum_{S_1} e^{\beta J S_1 S_2} e^{\beta H S_1} e^{\beta J S_1 S_0} \\ &= e^{\beta J S_2} e^{\beta H} e^{\beta J S_0} + e^{-\beta J S_2} e^{-\beta H} e^{-\beta J S_0} \end{aligned} \tag{4}$$

It is clear that, to go further, we need to be more systematic about how we add each successive spin. To do this, I will introduce a 2-dimensional Hilbert space with the basis

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

Then Z_1 is a matrix element computed for a matrix on this space,

$$Z_1 = \langle S_1| \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} |S_0\rangle \tag{6}$$

The expression for Z_2 is given systematically by computing the matrix product

$$Z_2 = \langle S_2| \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \begin{pmatrix} e^{\beta H} & 0 \\ 0 & e^{-\beta H} \end{pmatrix} \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} |S_0\rangle \tag{7}$$

and now the generalization is clear. Let

$$T_J = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} , \qquad T_H = \begin{pmatrix} e^{\beta H} & 0 \\ 0 & e^{-\beta H} \end{pmatrix} . \tag{8}$$

Then

$$Z_N = \langle S_N| \left(T_J T_H\right)^N T_H^{-1} |S_0\rangle \tag{9}$$

Let me make a small technical adjustment. If we define

$$T = T_H^{1/2} T_J T_H^{1/2} = \begin{pmatrix} e^{\beta J + \beta H} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta H} \end{pmatrix} \tag{10}$$

then T is a Hermitian matrix. Using this matrix, we have

$$Z_N = \langle S_N| T_H^{-1/2} \, T^N \, T_H^{-1/2} |S_0\rangle \ . \tag{11}$$

T is called the *transfer matrix*. Multiplying by T systematically adds one spin. For a system with periodic boundary conditions, there is an even simpler result,

$$Z_N = \mathrm{tr}[T^N] \tag{12}$$

If there is a Hilbert space, there must be a connection to quantum mechanics. In fact, it is very useful to borrow concepts from quantum mechanics to understand this formula. In quantum mechanics, time evolution is described by the operator

$$U(t) = e^{-it\mathcal{H}} \tag{13}$$

If we divide the (continuous) time into N discrete steps of size $\epsilon = t/N$, the discrete representation of time evolution is

$$\langle a| e^{-it\mathcal{H}} |b\rangle = \langle a| \left(e^{-i\epsilon\mathcal{H}}\right)^N |b\rangle \ . \tag{14}$$

In quantum statistical mechanics, the partition function is written

$$Z = \mathrm{tr}\left[e^{-\beta\mathcal{H}}\right] \tag{15}$$

which we might view as a trace over an evolution in imaginary time

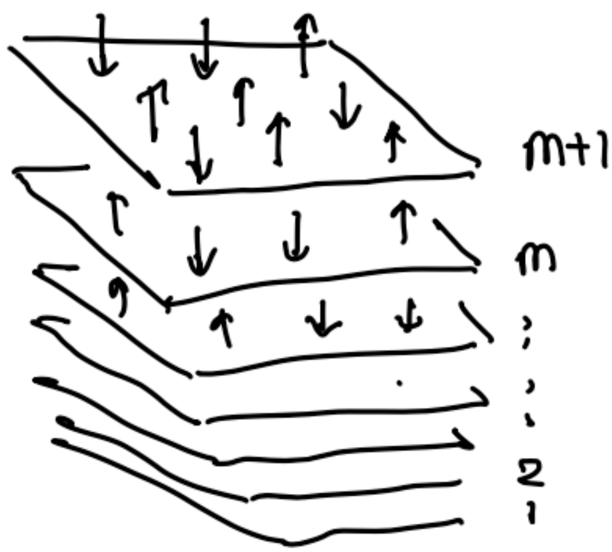
$$t = -i\beta \ . \tag{16}$$

If we divide this imaginary time interval into N pieces, $\epsilon = \beta/N$, our expression for Z becomes imaginary time evolution in discrete steps. Then

$$Z = \mathrm{tr}\Big[T^N\Big] \qquad \text{with} \qquad T = e^{-\epsilon\mathcal{H}} \tag{17}$$

The 1-d Ising model is a model in *classical* statistical mechanics, but we see that in this model the same structure appears. The partition function has the form (17) with the transfer matrix given in (10). We can then use our intuition from quantum mechanics to understand the behavior of a classical statistical system. Notice that this analogy relates classical statistical mechanics in 1 dimension to the quantum mechanics of finite systems (here, a system of 1 spin).

You can easily imagine a generalization of the concept of a transfer matrix to d-dimensional classical systems. If the system has M layers, we can add one more layer using a transfer matrix formalism



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The transfer matrix is now a very large matrix, so it is challenging to work with. For a model with N spins in each layer, the Hilbert space has dimension 2^N. Nevertheless, the expression for the free energy is the same

$$Z_N = \mathrm{tr}[T^N] \tag{19}$$

Through the logic above, the transfer matrix of a d-dimensional classical statistical mechanics problem is related to the Hamiltonian of a $(d-1)$-dimensional quantum mechanics problem.

Let's now get back to concrete calculations and work out the free energy of the 1-dimensional Ising model. If T is a Hermitian matrix, we can diagonalize it,

$$T = \mathcal{U} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \mathcal{U}^{-1} \tag{20}$$

Then

$$Z = \lambda_0^N + \lambda_1^N \ . \tag{21}$$

In the thermodynamic limit $N \to \infty$, only the largest eigenvalue of T contributes. This eigenvalue, through the quantum mechanics analogy, corresponds to the lowest energy state of the quantum mechanical Hamiltonian. We then find

$$Z = \lambda_0^N \tag{22}$$

and

$$F = -\frac{N}{\beta} \log \lambda_0 \tag{23}$$

Let's work this out explicitly for the 1-dimensional Ising model at $H = 0$. We have for this case

$$T = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \tag{24}$$

This matrix has the eigenvectors and eigenvalues

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \lambda_0 &= e^{\beta J} + e^{-\beta J} = 2\cosh \beta J \\ |\psi_1\rangle & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \lambda_1 &= e^{\beta J} - e^{-\beta J} = 2\sinh \beta J \end{aligned} \tag{25}$$

So, from the formulae above,

$$F = -\frac{N}{\beta} \log(2\cosh \beta J) \ . \tag{26}$$

If you look at this carefully, you might suspect that I have cheated you in my discussion of mean field theory. F is supposed at have a singularity at $T = T_c$, but this function is completely nonsingular all the way down to $T = 0$.

Before I discuss this point further, let's make a few more observations about this model. The Z_2 symmetry of the Ising model wtih $H = 0$ has implications for the eigenvectors. On this system, Z_2 is represented by operators $\{\mathbf{1}, P\}$, where P is the operator with the action

$$P\,|\uparrow\rangle = |\downarrow\rangle \ , \qquad P\,|\downarrow\rangle = |\uparrow\rangle \ , \tag{27}$$

so that $P^2 = 1$. On the 2×2 space, P has the matrix representation

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{28}$$

Then

$$P\,|\psi_0\rangle = (+1)\ \ |\psi_0\rangle \qquad P\,|\psi_0\rangle = (-1)\ \ |\psi_0\rangle\ \ . \tag{29}$$

We can use this observation to compute the local spin-spin correlation function in this model. Introduce an operator that measures the spin on a site.

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{30}$$

Notice that

$$P\mathcal{S} = -\mathcal{S}P\ , \tag{31}$$

that is, $\mathcal{S}$ reverses the P quantum number.

To compute the spin-spin correlation function, we insert the operator $\mathcal{S}$ to measure the spin value on the designated sites. Let $J > I$. Then

$$\langle S_J S_I \rangle = \frac{1}{Z}\mathrm{tr}\Big[T^{N-J}\mathcal{S}T^{J-I}\mathcal{S}T^{I}\Big] \tag{32}$$

or

$$\langle S_J S_I \rangle = \frac{1}{Z}\mathrm{tr}\Big[T^{N-(J-I)}\mathcal{S}T^{J-I}\mathcal{S}\Big]\ . \tag{33}$$

We would like to evaluate this quantity for $J - I|$ finite in the limit- $N \to \infty$. In this limit, the quantity $T^{N-(J-I)}$ is dominated by the largest eigenvalue of T, which has $Z_2 = +1$. The spin operator transforms the state with $P = +1$ to one with $P = -1$. In particular,

$$\mathcal{S}\,|\psi_0\rangle = |\psi_1\rangle \qquad \mathcal{S}\,|\psi_1\rangle = |\psi_0\rangle\ \ . \tag{34}$$

We then find

$$\langle S_J S_I \rangle = \frac{1}{\lambda_0^N}\mathrm{tr}\Big[\lambda_0^{N-(J-I)}\lambda_1^{(J-I)}\Big] = \left(\frac{\lambda_1}{\lambda_0}\right)^{(J-I)}\ . \tag{35}$$

that is

$$\langle S_J S_I \rangle = (\tanh\beta J)^{(J-I)}\ . \tag{36}$$

Since $|\tanh\beta J| < 1$ for all β, this is an exponentially decaying function for all nonzero temperatures. Then, by the logic explained in the previous lecture, the magnetization is zero at all nonzero temperatures and the system never magnetizes. Writing (36) in the form

$$\langle S_J S_I \rangle = \exp[-|J - I|/\xi(T)]\ , \tag{37}$$

we identify the *correlation length* $\xi(T)$ as

$$\xi(T) = 1/\log\tanh\beta J \tag{38}$$

This becomes extremely long at low temperatures

$$\xi(T) \sim \frac{1}{2} e^{\beta J} \tag{39}$$

as $\beta \to \infty$, but the correlation length is finite for any finite β. There is no phase transition.

This exact result would seem to shake up the intuition that I have been trying to build in the previous two lectures. However, thinking a little harder, there are reasons why the physics in 1 dimension should be special. In the mean-field picture, we expect that each spin is supported by its neighboring sites, such that, if the majority of sites around it have (say) spin up, that spin would also prefer to have spin up. However, in 1 dimension, a spin has only 2 neighbors. If the one to the left flips down by a thermal fluctuation, the system loses all memory of the ordered spins to the left

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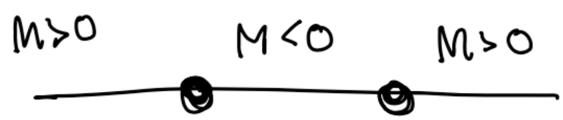
Then the spins to the right can flip down one by one.

The result that there are no order-disorder phase transitions in 1 dimension (if all interactions are short-ranged) is due to Lev Landau and is explained in the last section of Landau and Lifshitz, *Statistical Physics*. Here is the explanation given there: Consider a system with possible broken symmetry phases, here, spin up and spin down. In principle, the system can break up into n domains, each of which has a different value of the order parameter.



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The energy cost of forming these domains is

$$E \sim \Delta E \cdot n \tag{42}$$

However, the entropy gain is

$$S \sim n \log n \tag{43}$$

At any nonzero temperature, for large enough n, entropy always wins out over energy and the system becomes macroscopically disordered.

Before we leave the 1-dimensional Ising model, I would like to note that the results we found from the transfer matrix idea generalize to higher dimensions. As I have discussed above, for a system with only local interactions, we can always cut through the system perpendicular to any axis and consider the system as a succession of layers building up that direction. This is shown in (18). It leads in the same way as for the 1-dimensional Ising model to the result

$$Z = \mathrm{tr}[T^N] = (\lambda_0)^N \tag{44}$$

where N is the number of layers and λ_0 is the largest eigenvalue of the transfer matrix. I will construct T explicitly for the 2-dimensional Ising model in the next lecture.

To argue intuitively about the properties of the transfer matrix, recall from the above that

$$T = \exp[-\epsilon\mathcal{H}] \tag{45}$$

where $\mathcal{H}$ is the Hamiltonian of the analogous quantum mechanical problem. Then the vector with the largest eigenvalue of T is the ground state of $\mathcal{H}$ and, similar, smaller eigevalues of T arise from higher-energy states of $\mathcal{H}$.

For $H = 0$, the d-dimensional Ising model continues to have a Z_2 symmetry, and this symmetry is reflected as a symmetry of the transfer matrix,

$$[T, P] = 0 \tag{46}$$

The largest eigenvalue of T will belong to the $P = +1$ sector. Since the action of P is to flip all spins, we continue to have the relation

$$P\mathcal{S}_{\mathcal{I}} = -\mathcal{S}_{\mathcal{I}}P \tag{47}$$

between P and the operator S_I that measures the spin at the site I. Then, again, the operator S_i turns a state in the $P = +1$ sector into a state in the $P = -1$ sector. Using this information, we can calculate the spin-spin correlation function just as we did in (35). Since the Hilbert space is now bigger, we cannot obtain so exact a result, but still it follows that the large-distance behavior of the spin-spin correlation function is

$$\langle S_J S_I \rangle = c\Big(\frac{\lambda_1}{\lambda_0}\Big)^{(J-I)} \tag{48}$$

where λ_0 is the largest eigenvalue of T, in the sector $P = +1$, and λ_1 is the largest eigenvalue of T in the sector $P = -1$. Thus, it is true very generally that $\langle S_J S_I \rangle$ decays exponentially at large $|J - I|$, unless there is a degeneracy $\lambda_0 = \lambda_1$.

A similar result will follow for a statistical mechanical system with any higher symmetry. Let Q be the charge corresponding to the symmetry. Then λ_0, the largest eigenvalue of T, corresponds to the ground state of $\mathcal{H}$, which lies in a state with $Q = 0$, and λ_1 corresponds to a state of higher energy in $\mathcal{H}$ that belongs to a sector of nonzero Q.

To obtain spontaneous symmetry breaking, we need an exact degeneracy between states with zero and nonzero charge. This actually fits very nicely with the picture of different superselection sectors. To describe this concretely, let's return to the Ising case. If there is spontaneous symemtry breaking with superselection sectors $\mathcal{S}_+$ and $\mathcal{S}_-$, let $|0,+\rangle$ and $|0,-\rangle$ be the ground states of $\mathcal{H}$ in each selection sector. By symmetry, these states have the same $\mathcal{H}$ eigenvalue λ. Then the states

$$|0\rangle = \frac{1}{\sqrt{2}}(|0,+\rangle + |0,-\rangle) \quad \text{and} \quad |1\rangle = \frac{1}{\sqrt{2}}(|0,+\rangle - |0,-\rangle) \tag{49}$$

will be degenerate states with the same eigenvalue λ, the first in the $P = +1$ sector, the second in the $P = -1$ sector.

In this language, it is clear why there cannot be spontaneous symmetry breaking in 1 dimesion. The transfer matrix T or the Hamiltonian $\mathcal{H}$ acts on a Hilbert space with only a finite number of states. This Hilbert space is just not big enough to contain states that cannot be connected by any local operator.

As we continue in the course, we will see more examples in which the behavior of a statistical mechanical system depends crucially on the spatial dimensionality. We will speak of a *lower critical dimensionality* d_ℓ such that there is no ordered state for $d \leq d_\ell$. For the Ising model, $d_\ell = 1$. Our arguments show that, for any system with any local interactions, there cannot be a phase transition in 1 dimension. However, it is possible that, for magnets with high symmetry, the lower critical dimensionality can be at a higher value of d. Later we will also encounter an *upper critical dimensionality* that signals another change of behavior depending on the dimensionality.