Oct. 14

Physics 212 – Statistical Mechanics

Supplement: Green's Functions and Function Spaces

In the next lecture there is a moment when I claim:

> The inverse of the operator $(-\nabla^2 + A)$ is the Green's function that solves the equation $(-\nabla^2 + A)G(x, y) = \delta(x, y)$.

It seems that many people in the class are not familiar with the notion of a "Green's function", much less the notion of the inverse of an operator. This is a shame, because these are very useful concepts. In fact, the first things that a physicist is supposed to do when confronted with a set of mathematical equations for an unknown system are (1) linearize then, and (2) compute the Green's function. This typically gives a large amount of insight that one can use in planning a more complete analysis of the problem.

Thus, these notes fill in some background on Green's functions. It is very important that you completely absorb all of the lessons about Green's functions given in this note.

I also try to explain further the statement above that the Green's function is the inverse of the associated operator. This discussion is somewhat abstract, but it is actually useful to work at this level of abstraction, not only for quantum mechanics but also for the general study of systems governed by partial differential equations. I hope that the discussion here will make the abstractions introduced in the course more clear to you.

Basic definitions

Consider a linear system for a set of fields $\phi(x)$ governed by the differential equation

$$\mathcal{O}\ \phi(x) = 0\ , \tag{1}$$

where $\mathcal{O}$ is a linear differential operator. We might also consider the inhomogeneous version of this equation,

$$\mathcal{O}\ \phi(x) = \rho(x) \tag{2}$$

To analyze these equations, it is very useful to consider the *Green's function* of $\mathcal{O}$, defined as the solution to the differential equation

$$\mathcal{O}\ G(x, y) = \delta(x - y) \tag{3}$$

where $\delta(x)$ is the Dirac delta function. The Green's function gives the behavior of the system, measured at x, due to a perturbation placed at the point y. Thus, this function encodes the complete mechanical response of the linear system. Since the system is linear, we can superpose the responses to perturbations placed at different points to evaluate any general mechanical motion of the system.

In particular, the solution to (2) is

$$\phi(x) = \int dy \; G(x,y) \; \rho(y) \tag{4}$$

To show this, just act the linear operator $\mathcal{O}$ on both sides of this equation. You will find

$$\mathcal{O} \; \phi(x) = \int dy \; \delta(x-y) \; \rho(y) = \rho(x) \; . \tag{5}$$

Laplace equation

Let me give you some simple examples: First, consider the Laplace equation in 3-dimensional electrostatics. We would like to find the electrostatic potential $\varphi(x)$ that results from a charge distribution $\rho(x)$. This is the solution to the equation

$$(-\nabla^2) \; \varphi(x) = \rho(x) \; . \tag{6}$$

Let $G(x,y)$ be the solution to the associated Green's function equation

$$-\nabla_x^2 \; G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}) \; . \tag{7}$$

Here and below, the subscript x means that the derivatives in the operator act on x. You know that the solution to this equation in unbounded space is the Coulomb potential

$$G(\vec{x},\vec{y}) = \frac{1}{4\pi}\frac{1}{|x-y|} \; . \tag{8}$$

Then the solution to the inhomogeneous equation (6) is

$$\varphi(\vec{x}) = \int d^3y \; G(\vec{x},\vec{y}) \; \rho(\vec{y}) \tag{9}$$

or, more explicitly,

$$\varphi(\vec{x}) = \int d^3y \; \frac{1}{4\pi}\frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|} \; . \tag{10}$$

If we are not in unbounded space but rather in some region with boundary conditions, the method of solution still holds. For example, if there is a conducting plane at $z=0$ such that $\phi(\vec{x}) = 0$ on this plane, then the Green's function that solves Laplace's equation with the given boundary conditions is

$$G(\vec{x},\vec{y}) = \frac{1}{4\pi}\Big[\frac{1}{|x-y|} - \frac{1}{|x-\overline{y}|}\Big] \; , \tag{11}$$

where, if $\vec{y} = (x, y, z)$, $\overline{y}$ is the image source at $(x, y, -z)$. In complete generality, if we can solve for the Green's function with the given boundary conditions, the solution (9) will solve the inhomogeneous equation (6) and will automatically have the correct boundary conditions.

Harmonic oscillator equation

Here is another example that illustrates some different features. Consider a damped harmonic oscillator obeying the equation

$$\left(m\frac{d^2}{dt^2} - m\Omega^2 + m\gamma\frac{d}{dt}\right)x(t) = 0 \tag{12}$$

The Green's function for this equation is the solution to the equation

$$\left(m\frac{d^2}{dt^2} + m\Omega^2 + m\gamma\frac{d}{dt}\right)G(t, t_0) = \delta(t - t_0) \tag{13}$$

However, probably I should not say "the" Green's function, because the form of the Green's function will depend on the boundary conditions imposed in time. For this problem, the corresponding inhomogeneous equation is

$$\left(m\frac{d^2}{dt^2} + m\Omega^2 + m\gamma\frac{d}{dt}\right)x(t) = F(t) \ , \tag{14}$$

where $F(t)$ is a force applied as a function of time. Physically, we would always like the response to occur after the force is applied. This is simple causality. We can implement the requirement of causality by choosing the Green's function in (13) to satisfy

$$G(t, t_0) = 0 \text{ for all } t < t_0 \ . \tag{15}$$

For $\gamma = 0$, consider the function

$$G(t, t_0) = \begin{cases} 0 & t < t_0 \\ \frac{1}{m\Omega}\sin\Omega(t - t_0) & t > t_0 \end{cases} = \frac{1}{m\Omega}\sin\Omega(t - t_0)\ \theta(t - t_0) \ , \tag{16}$$

where $\theta(t)$ equals 1 for $t > 0$ and 0 for $t < 0$. It is easy to see that this equation satisfies (15) and satisfies (13) for any $t \neq t_0$. Just near t_0, we need one more piece of analysis. Integrate (13) over a small region around $t = t_0$:

$$\int_{t_0-\epsilon}^{t_0+\epsilon} dt\ \left(m\frac{d^2}{dt^2} + \cdots\right) = \int_{t_0-\epsilon}^{t_0+\epsilon} dt\ \delta(t - t_0) = 1 \tag{17}$$

Then

$$m\frac{d}{dt}G(t_0 + \epsilon, t_0) - m\frac{d}{dt}G(t_0 - \epsilon, t_0) = m\frac{d}{dt}G(t_0 + \epsilon, t_0) - 0 = 1 \ , \tag{18}$$

which is satisfied by (16). Then the general solution to the homogeneous problem taking into account the requirement of causality is

$$x(t) = \int_{-\infty}^{\infty} dt_0 \; \frac{F}{m\Omega} \sin\Omega(t-t_0) \; \theta(t-t_0)$$
$$= \int_{-\infty}^{t} dt_0 \; \frac{F}{m\Omega} \sin\Omega(t-t_0) \; . \tag{19}$$

This solution is the superposition of the solutions generated from small increments of force $F(t_0)$ at all times earlier than t.

The function (16) is called the *retarded Green's function.* The opposite case, in which the response occurs entirely before the perturbation, is called the *advanced Green's function.*

Harmonic oscillator equation – 2nd method

It is not so hard to construct the Green's function and the general solution for any positive value of γ by straightforwardly solving the equation (13). However, I would like to use a different method which illustrates yet more features of the Green's function. Here I will construct the Green's function by Fourier transformation,

$$x(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega) \qquad \tilde{x}(\omega) = \int dx e^{i\omega t} x(t) \tag{20}$$

By translation invariance, $G(t, t_0) = G(t - t_0)$, so it suffices to find the Green's function for $t = 0$. For $t_0 = 0$, the right-hand side of (13) is $\delta(t)$. Note that

$$\int dt e^{i\omega t} \delta(t) = 1 \tag{21}$$

so the Fourier transform of (13) for $t_0 = 0$ is

$$m(-\omega^2 + \Omega^2 - i\gamma\omega)\tilde{x}(\omega) = 1 \; . \tag{22}$$

and the solution to the equation (13) is found by solving this and inverting the Fourier transform

$$G(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{-1/m}{(\omega^2 - \Omega^2 + i\omega\gamma)} \tag{23}$$

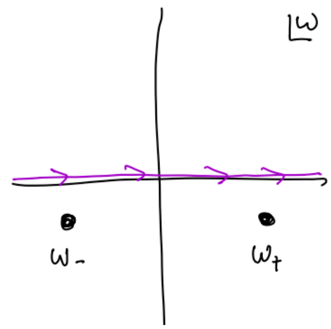
This is a tricky but very compelling integral. It is useful to think of ω as a complex variable. The integrand has poles at the solutions of

$$\omega^2 - \Omega^2 + i\omega\gamma = 0 \; , \tag{24}$$

which occur at

$$\omega_\pm = -i\gamma/2 \pm [\Omega^2 + \gamma^2/4]^{1/2} \tag{25}$$

Notice that the poles are in the lower half ω plane. The integral is along the real axis in this plane. So if we can push the coutour upward in the plane to infinity, we will not encounter any poles. However, if we push the contour downward to infinity, it will stick on the poles,



(26)

Consider first doing the integral for $t < 0$ ($t < t_0$). Then if we move the contour upward so that $\omega \to \omega + i\eta$, $\eta > 0$,

$$e^{-i\omega t} \to e^{-i\omega t} e^{+\eta t} \ , \tag{27}$$

and the second term has a negative number in the exponent. Then the further we push the contour upward, the smaller is the integrand. Since the value of the integral is independent of the contour position as long as we do not cross any singularities, we find

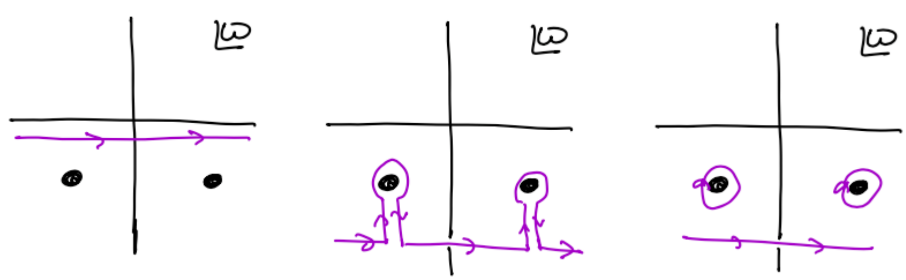
$$G(t) = 0 \text{ for all } t < 0 \ . \tag{28}$$

This argument in the complex plane automatically implements causality.

Next, consider doing the integral for $t > 0$ ($t > t_0$). In this case, we would like to make the integrand smaller by moving the contour down, $\omega \to \omega - i\eta$, $\eta > 0$. In this case,

$$e^{-i\omega t} \to e^{-i\omega t} e^{-\eta t} \ , \tag{29}$$

Again, the integrand gives 0 when pushed off to infinity, but in this case we must also pick up the contributions from the two poles.



(30)

These are

$$(-2\pi i)\frac{1}{2\pi}(-1/m)\left[\frac{1}{(2i(\Omega' - i\gamma/2)}e^{-i\Omega' t}e^{-\gamma t/2} + \frac{1}{(-2i(\Omega' + i\gamma/2)}e^{+i\Omega' t}e^{-\gamma t/2}\right] \ , \tag{31}$$

where

$$\Omega' = [\Omega^2 + \gamma^2/4]^{1/2} \tag{32}$$

The result is

$$G(t) = \frac{1}{m[\Omega'^2 + \gamma^2/4]}[\Omega' \sin \Omega' t - \gamma/2 \cos \Omega' t] e^{-\gamma t/2} \ . \tag{33}$$

This is amazing. The fact that the poles are located in the lower half ω plane insures causality. The locations of the poles give the response frequencies of the system: The real parts of the poles give the oscillation frequencies, and the imaginary parts give the rates of damping. Everything we would like to know about the linear response of the system is encoded in the Green's function! And, remarkably, if we can have access to the Fourier transform of the Green's function, we can see it all more clearly.

Do I need to tell you that these remarks generalize to every linear system evolving in time? This is true, so please master this derivation.

Function space

Now I would like to introduce the other property of the Green's function, that it is the inverse of its defining operator on the space of functions. By "the space of functions", I mean an infinite dimensional space that is controlled in a well-defined mathematical sense. To do 1-particle quantum mechanics, we restrict the Schrödinger wavefunction $\psi(x)$ to be a square-integrable function of x. Here I will assume that the function space is properly defined to be a Hilbert space on which appropriate linear operators can act.

In quantum mechanics, we write transformation of wavefunctions by linear operators in the form

$$|h\rangle = F\,|k\rangle \ . \tag{34}$$

Writing this out in terms of the Schrödinger wavefunctions describing the states, this equation would be

$$h(x) = \int dy \ F(x,y) \ k(y) \tag{35}$$

I would like you to think of this linear operation as matrix multiplication in the function space. Think of x and y as indices of the matrix $F(x,y)$ or of the vectors $h(x)$ and $k(y)$. The matrix multiplication is accomplished by summation over one index of the matrix. You can make this correspondence precise by approximating space by a lattice of points or considering $h(x)$ and $k(y)$ to be piecewise constant functions. Eventually, we will take the limit in which $h(x)$ and $k(y)$ become arbitrarily varying (*e.g.*, square-integrable) functions on a continuum.

The composition of two linear operators F and G gives

$$h(x) = \int dw\, dy \ F(x,w) \ G(w,y) \ k(y) \ . \tag{36}$$

or

$$(F\ G)(x,y) = \int dw\ F(x,w)\ G(w,y)\ . \tag{37}$$

This is the equation for the multiplication or composition of two matrices on the function space.

We can include differential operators into this picture. For example, for the Laplace operator, let

$$L(x,y) = -\nabla^2\ \delta(x-y) \tag{38}$$

Then

$$\int dy\ L(x,y)\ k(y) = -\nabla_x^2\ k(x) \tag{39}$$

is the linear action of the Laplace operator on the function (or vector) $k(x)$.

The identity operator on this space is

$$I(x,y) = \delta(x-y) \tag{40}$$

To see that this is true, consider that $I\,|k\rangle$ is represented by

$$\int dy\ I(x,y)\ k(y) = \int dy\ \delta(x-y)\ k(y) \tag{41}$$

so, finally

$$I\,|k\rangle = |k\rangle \tag{42}$$

From this point of view, we can see that the equation

$$-\nabla^2\ G(x,y) = \delta(x-y) \tag{43}$$

takes the form in the the function space as

$$\int dy L(x,w)\ G(w,y) = I(x,y)\ , \tag{44}$$

that is

$$L\ G = I\ , \tag{45}$$

Thus, the Green's function of the Laplace operator is indeed the inverse of the Laplace operator as a linear operator on the function space.

Quadratic forms and Gaussian correlations

A quadratic form involving the Laplace operator has the form

$$\int dx \Big\{ m(x)(-\nabla^2)m(x) \Big\} = \langle m|\,L\,|m\rangle = \int dx\,dy\ m(x)\ L(x,y)\ m(y)\ . \tag{46}$$

I hope that it is clear in this notation that this equation is the direct generalization to an infinite-dimensional function space of the quadratic form in a finite dimensional space

$$\sum_{a,b} m_a M_{ab} m_b \tag{47}$$

Let's explore this correspondence a little more. I would like to compute the Gaussian integral

$$\mathcal{I} = \int dm^a \ \exp[-\frac{1}{2}\ m_a M_{ab} m_b] \tag{48}$$

where the sum over repeated indices now understood. We can solve this integral by going to a basis of eigenvectors of the matrix M. These are given by

$$\sum_b M_{ab}\xi^j_b = \lambda_j \xi^j_b \tag{49}$$

We can represent the m_a in this basis as a linear combination of normalized eigenvectors,

$$m_a = c_j \xi^j_a \tag{50}$$

The quadratic form becomes

$$\sum_{a,b} m_a M_{ab} m_b = \sum_j \lambda_j c_j^2 \tag{51}$$

and we can carry out the integral over all m_a by integrating over all values of c_j. The result is

$$\mathcal{I} = \prod_j \Big(\frac{2\pi}{\lambda_j}\Big)^{1/2} \tag{52}$$

The expectation value $\langle m_a m_b\rangle$ can be evaluated similarly. First

$$\int dc_j \ c_k c_\ell \exp[-\frac{1}{2}\lambda_j c_j^2] = \frac{1}{\lambda_j}\delta_{jk} \cdot \prod_j \Big(\frac{2\pi}{\lambda_j}\Big)^{1/2} . \tag{53}$$

Then

$$\begin{aligned} \int dm_a \ m_c m_d \ \exp[-\frac{1}{2}\ m_a M_{ab} m_b] &= \xi^j_c \xi^k_d \cdot \frac{1}{\lambda_j}\delta_{jk} \cdot \prod_j \Big(\frac{2\pi}{\lambda_j}\Big)^{1/2} \\ &= \xi^j_c \frac{1}{\lambda_j} \xi^j_d \cdot \prod_j \Big(\frac{2\pi}{\lambda_j}\Big)^{1/2} . \end{aligned} \tag{54}$$

We can recognize the first term as a representation of the inverse matrix to the matrix M_{ab}. Dividing by (52),

$$\langle m_c m_d \rangle = (M^{-1})_{cd} \ . \tag{55}$$

By analogy, we can evaluate the integral of

$$\exp\Big[-\int dx \Big\{m(x)(-\nabla^2)m(x)\Big\}\Big] \tag{56}$$

over the whole space of functions $m(x)$. We have seen that the expression in the exponent is

$$\int dx\, dy\ m(x)\ L(x,y)\ m(y)\ . \tag{57}$$

The eigenvalues of the linear operator, in this case, the Laplacian, satisfy

$$-\nabla^2\ \xi^j(x) = \int dy\ L(x,y)\ \xi^j(y) = \lambda_j\ \xi^j(x)\ . \tag{58}$$

We can expand any function $m(x)$ in a basis of normalized eigenfunctions

$$m(x) = \sum_j c_j \xi^j(x). \tag{59}$$

Then the quadratic form (46) becomes

$$\langle m|\, L\, |m\rangle = \int dx\, dy\ m(x)\ L(x,y)\ m(y) = \sum_j \lambda_j c_j^2 \tag{60}$$

where now the sum over j runs over an infinite number of eigenfunctions with increasing eigenvalues. Typically, though, we would regularize this system by ignoring eigenvalues with $\lambda_j > \Lambda$, giving a finite-dimensional problem, and then, at the end of the argument, taking the limit $\Lambda \to \infty$. In the regulated case, for any finite number of eigenvalues, the calculation of the correlation function would proceed just as in the finite-dimension problem with the quadratic form M_{ab}. We then find

$$\langle m(x)m(y)\rangle = \sum_j \xi^j(x)\frac{1}{\lambda_j}\xi^j(y)\ . \tag{61}$$

The differential equation for the Green's function is

$$-\nabla_x^2\ G(x,y) = \int dw L(x,w)G(w,y) = \delta(x-y) \tag{62}$$

Expand $G(x,y)$ in a basis of the eigenfunctions of the operator $L(x,y)$

$$G(x,y) = \sum_j \xi^j(x)g^j(y) \tag{63}$$

and use the completeness relation

$$\delta(x-y) = \sum \xi^j(x)\xi^j(y) \tag{64}$$

Then the equation (62) becomes

$$\lambda_j \xi^j(x) g^j(y) = \xi^j(x)\xi^j(y) \tag{65}$$

and so

$$G(x,y) = \sum_j \xi^j(x) \frac{1}{\lambda_j} \xi^j(y) \tag{66}$$

Then, finally,

$$\langle m(x) m(y) \rangle = G(x,y) \ . \tag{67}$$

In this derivation, actually, I did not use any property of the Laplace operator other than the fact that it is a Hermitian operator. So, actually, (67) holds for any Hermitian operator $L(x,y)$.

This analysis suggests another definition. For F a linear operator, we can define the *functional determinant* of F as

$$\det F = \prod_j \lambda_j \ , \tag{68}$$

where λ_j are the eigenvalues of F. Typically $\det F$ is infinite, but it can be regulated, and that regulated function is often useful in quantum-mechanical problems.

The idea of the Green's function as the inverse of a differential operator comes up often in quantum mechanics, particularly in scattering theory. There is it common to represent the solution of equations

$$[\frac{p^2}{2m} + V] \, |\psi\rangle = |\phi\rangle \tag{69}$$

as

$$\frac{1}{[p^2/2m + V]} \, |\phi\rangle \quad \text{or} \quad [p^2/2m + V]^{-1} \, |\phi\rangle \ . \tag{70}$$

It doesn't matter that the computation of the inverse Schrödinger operator involves the solution of a complicated partial differential equation. This notation still clarifies one's thinking.

Alternative proof the Gaussian integral formula

If you are uncomfortable with the analogy I am using between finite- and infinite-dimensional spaces, I have that the following proof of the formula for the correlation function of Gaussian random variables will make you happier. This argument also uses the fact that the Green's function is the inverse of the associated linear operator, but maybe in a more intuitively acceptable way.

Let's consider the finite-dimensional case first. We wish to compute the correlation function

$$\langle m_c m_d \rangle = \int d^n m \exp[-\sum_{ab} \frac{1}{2} \int dx \ m_a \ M_{ab} m_b] \ m_c m_d$$

$$/ \int d^n m \exp[-\sum_{ab} \frac{1}{2} \int dx \ m_a \ M_{ab} m_b] \ . \tag{71}$$

Let $Z(j)$ be the following integral

$$Z(j) = \int d^n m \exp[-(\sum_{ab} \frac{1}{2} \int dx \ m_a \ M_{ab} m_b - \sum_a m_a j_a)] \ , \tag{72}$$

o that Z, the denominator of (82), is given by $Z(j)$ evaluated with $j_a = 0$ for all a. Notice that the derivative of $Z(j)$ is

$$\frac{\partial}{\partial j_c} Z(j) = \int d^n m \exp[-(\sum_{ab} \frac{1}{2} \int dx \ m_a \ M_{ab} m_b - \sum_a m_a j_a)] \ m_c \ . \tag{73}$$

Similarly, differentiating with respect to j_d for multiple values of d gives the numerator of a correlation function of m_d's. In particular,

$$\frac{1}{Z} \frac{\partial^2}{\partial j_c \partial j_d} Z(j) \Big|_{j=0} = \langle m_c m_d \rangle \ . \tag{74}$$

We can actually compute $Z(j)$ explicitly by making a change of variables. Let

$$m'_a = m_a - (M^{-1})_{ab} j_b \ . \tag{75}$$

This is a simple shift of the integration variables, so

$$d^n m' = d^n m \tag{76}$$

Substituting m'_a for m_a,

$$\begin{aligned}
\frac{1}{2} m_a M_{ab} m_b - m_a j_a &= \frac{1}{2}(m'_a + (M^{-1})_{ac} j_c) M_{ab} (m'_b + (M^{-1}_{bd} m_d) - (m'_a + (M^{-1})_{ac} j_c) j_a \\
&= \frac{1}{2}(m'_a M_{ab} m'_b + m'_a M_{ab} (M^{-1})_{bd} m_d + j_c (M^{-1})_{ca} M_{ab} m'_b \\
&\qquad + j_c (M^{-1})_{ca} M_{ab} (M^{-1})_{bd} j_d) - m'_a j_a - j_c (M^{-1})_{ca} j_a \\
&= \frac{1}{2}(m'_a M_{ab} m'_b + 2 m'_a j_a + j_c (M^{-1})_{ca} j_a) - m'_a j_a - j_c (M^{-1})_{ca} j_a \\
&= \frac{1}{2} m'_a M_{ab} m'_b - \frac{1}{2} j_c (M^{-1})_{ca} j_a
\end{aligned} \tag{77}$$

Thus,

$$Z(j) = \int d^n m' \exp[-(\sum_{ab} \frac{1}{2} \int dx \ m'_a \ M_{ab} m'_b) + (half \sum_{ab} j_a (M^{-1})_{ab} j_b)] \ . \tag{78}$$

We recognize that this expression contains the expression for Z, so

$$Z(j)/Z = \exp[\frac{1}{2} \sum_{ab} j_a (M^{-1}_{ab} j_b] \tag{79}$$

where the integral of m' exactly cancels out.

Now we are almost done. Taking two derivatives of (79), we find

$$\frac{\partial^2}{\partial j_c \partial j_d} \frac{Z(j)}{Z} = \frac{Z(j)}{Z} \cdot \left[(M^{-1})_c d + \mathcal{O}(j_a^2) \right] . \tag{80}$$

Then, setting $j = 0$, we find

$$\langle m_c m_d \rangle = (M^{-1})_{cd} \ , \tag{81}$$

in accord with the result obtained in class.

To compute the correlation function

$$\langle m(y) m(z) \rangle = \int \mathcal{D}m \exp[-\frac{1}{2} \int dx \ m(x) \ \mathcal{O} \ m(y)] \ m(y) m(z)$$
$$/ \int \mathcal{D}m \exp[-\frac{1}{2} \int dx \ m(x) \ \mathcal{O} \ m(y)] \tag{82}$$

we can repeat these steps exactly for the infinite-dimensional integral. Let $Z[j]$ be the functional of $j(x)$

$$Z[j(x)] = \int \mathcal{D}m \exp[-\int dx \ (\frac{1}{2} m(x) \ \mathcal{O} \ m(x) - m(x) j(x))] \ . \tag{83}$$

so that Z, the denominator of (82), is given by $Z[j]$ evaluated with $j(x) = 0$. Notice that the variational derivative of $Z[j]$ is

$$\frac{\delta}{\delta j(y)} Z[j] = \int \mathcal{D}m \exp[-\int dx \ [\frac{1}{2} m(x) \ \mathcal{O} \ m(y) - m(x) j(x)] \ \ m(y) \ , \tag{84}$$

and, similarly, differentiating with respect to $j(z)$ for multiple points gives the numerator of a correlation function of $m(z)$'s. In particular,

$$\frac{1}{Z} \frac{\delta^2}{\delta j(y) \, \delta j(z)} Z[j] \Big|_{j(x)=0} = \langle m(y) m(z) \rangle \ . \tag{85}$$

We can evaluate $Z[j]$ explicitly using a change of variables similar to(75). Let

$$m(x) = m'(x) - \int dy G(x,y) m(y) \ , \tag{86}$$

where $G(x,y)$ is the Green's function for the operator $\mathcal{O}$. The quadratic form in the exponent of (83) is then

$$\int dx \ (\frac{1}{2} (m'(x) + \int dw \ G(x,w) j(w)) \ \mathcal{O} \ (m'(x) + \int dv \ G(x,v) j(v)$$
$$-(m'(x) + \int dw \ G(x,w) j(w)) \, j(x))$$
$$= \frac{1}{2} (\int dx \ m'(x) \, \mathcal{O} \, m'(x) + \int dx dw \ j(w) G(x,w) \mathcal{O}_x m'(x) + \int dw \ m'(x) \mathcal{O} \, G(x,w) j(w)$$
$$+ \int dv dw \ j(v) G(x,v) \mathcal{O} \, G(x,w) j(w)) - \int dx \ m'(x) j(x) - \int dx \ dw G(x,w) j(w) \tag{87}$$

We can simplify the third and fourth term here by using

$$\mathcal{O}_x \; G(x,w) = \delta(x-w) \tag{88}$$

For the second term, we use the fact that $\mathcal{O}$ is a self-adjoint operator,

$$\int dxdw \;\; j(w)G(x,w)\mathcal{O}_x m'(x) = \int dxdw \;\; j(w) \;\; \mathcal{O}_x G(x,w) m'(x) = \int dx \;\; m'(x) j(x) \tag{89}$$

The the second, third, and fifth terms cancel and we find the quadratic form becomes

$$\frac{1}{2}\int dx \;\; m'(x)\,\mathcal{O}\,m'(x) - \frac{1}{2}\int dxdw \;\; j(x)\,G(x,w)\,j(w) \tag{90}$$

This is just the analog of the last line of (77). Thus we find

$$Z[j] = Z \cdot \exp[\frac{1}{2}\int dxdw \;\; j(x)\,G(x,w)\,j(w)] \;. \tag{91}$$

Then it follows that

$$\langle m(y) m(z) \rangle = \frac{\delta^2}{\delta j(y)\,\delta j(z)} \frac{Z[j]}{Z}\bigg|_{j(x)=0} = G(y,z) \;. \tag{92}$$

In either the finite- or the infinite-dimensional case, try taking more derivatives with respect to j_a or $j(x)$ before setting $j = 0$. You will find that this gives another proof of Wick's theorem.