Physics 212 – Final Exam

(due Wednesday, December 14)

I will use the quizzes and the final in this course to assign grades, keeping in mind that anyone who hands in all of the problem sets is assured a reasonable grade in the course. Because this exam will be graded, please abide by these rules:

- The quizzes are open-book. You may use any reference resources that you find. You may use your mathematical software such as MatLab or Mathematica. However, please do not collaborate with other students or ask help from other people—except that, if you have any questions about the quiz, please feel free to email me (mpeskin@slac.stanford.edu).
- The final is posted at the course web site:

  `https://www.slac.stanford.edu/~mpeskin/Physics212/`

  If there are corrections, the latest version can be found there.
- If you plan not to hand in the exam, this is acceptable, but please email me and let me know.
- Please write on your solution: "I acknowledge the Stanford Honor Code." and sign it.
- Please hand in the final to Gradescope by 6:00pm on Wednesday, Dec. 14. This is a sharp deadline. If this poses a problem for you, please email me in advance.

The final will be worth 50 points. Partial credit will be given.

1. This problem concerns the theory of *percolation*. In particular, it involves *bond percolation*. Imagine that you have a d-dimensional lattice of sites, which you might think of as an empty circuit board. Now place wires with zero resistance connecting nearest neighbor sites. Place a wire on a given link with probability p; then there will be no wire connection with probability $(1-p)$. At small values of p, only small clusters of sites will be connected. However, at some value of p, there will be a cluster of infinite size, so that current can flow from one side of the circuit board to the other. This percolation. It is a critical phenomenon. The critical value of p is called p_c, and a variety of quantities related to the clusters will have non-analytic behavior as $p \to p_c$. We can analyze this system using methods from this course.

   (a) Consider the bond percolation problem in 1 dimension. Show that there is no percolation for any $p < 1$. Show that the probability per bond of being in a connected cluster of s bonds is

   $$\mathcal{P}(s) = p^s(1-p)^2 \tag{1}$$

   Check this result by showing that $(1-p) + \sum_{s=1}^{\infty} s\mathcal{P}(s) = 1$. Why is this the correct sum rule?

   (b) In higher dimensions, let $C(\vec{x}, \vec{y})$ be the probability that the points $\vec{x}$ and $\vec{y}$ are members of a common connected cluster. From our experience with magnetic systems, we would expect that this correlation function falls off exponentially for $p < p_c$,

   $$C(\vec{x}) \sim \exp[-|x|/\xi] \qquad \text{with} \qquad \xi(p) \sim (p_c - p)^{-\nu} \tag{2}$$

   and, at $p = p_c$, this correlation function behaves in a scale-invariant fashion

   $$C(\vec{x}, \vec{0}) \sim \frac{1}{|x|^{d-2+\eta}} \ . \tag{3}$$

   Then

   $$\mathcal{N}(p) = \sum_{\vec{x}} C(\vec{x}, \vec{0}) \tag{4}$$

   is the typical number of sites in a cluster connected to the point $\vec{0}$. Argue that $\mathcal{N}(p)$ diverges as $p \to p_c$ as

   $$\mathcal{N} \sim (p_c - p)^{-\gamma} \tag{5}$$

   and compute γ in terms of ν and η.

   (c) A way to look at the result of part (b) is to think in terms of a distribution of cluster sizes s, where s is the number of bonds in the cluster. Let $n(s)$ be the average number of clusters per site with s bonds. Since every bond is in a cluster of some size, we expect

   $$\sum_s s\, n(s) = p \ . \tag{6}$$

Argue that

$$\mathcal{N} = \sum_s s^2 n(s) \tag{7}$$

Notice that the sum (6) is convergent as $p \to p_c$ while the sum (7) diverges. For p just below p_c, $n(s)$ should have a scale-invariant form

$$n(s) \sim s^{-1/\sigma} \tag{8}$$

up to sizes $s^{1/d} \sim \xi$. Using this information, compute σ in terms of ν and η. [Note: for the purpose of the exam, this question slightly oversimplifies the real situation.]

(d) For a 2-dimensional square lattice of N sites, develop a series expansion for $\mathcal{N}(p)$ in powers of p. In doing this, it would be nice not to have to count the many links with no bonds. So set

$$q = \frac{p}{(1-p)} \tag{9}$$

Then we can assign the weight q to bonds and the weight 1 to links with no bonds. The expectation value $C(\vec{x}, \vec{0})$ is then given by the ratio in which the denominator counts all ways to place wires on the lattice links with weight q and the numerator counts the placements in which the links connect the site $\vec{x}$ to the site $\vec{0}$. Show that the denominator is

$$1 + 2Nq + N(2N-1)q^2 + \cdots \tag{10}$$

This series sums to $(1+q)^{2N} = (1/(1-p))^{2N}$, but it is useful to keep it in the form of the series (10) to cancel the "disconnected" terms that arise in computing the numerator. You may assume that the lattice is periodically connected and ignore effects of the faraway boundary.

(e) Work out the expansion of the numerator in powers of q including terms of order q^5. Show that the terms involving N cancel between numerator and denominator and find the series expansion for $\mathcal{N}(q)$ in powers of q.

(f) Show that, for the power series expansion of the singular function $(1 - x/a)^{-\gamma}$ in powers of x, the ratio of successive series coefficients

$$a_n/a_{n-1} = \frac{(\gamma + n - 1)}{n} \frac{1}{a} \tag{11}$$

Using this formula and the data from part (e) to estimate q_c (and thus p_c) and the exponent γ.

(g) For the 2-dimensional square lattice, there is a duality that relates the percolation problem on the lattice to the percolation problem on the dual lattice. Use this duality and the assumption that there is only one phase transition to find p_c for this case.

(h) Construct an approximate recursion formula for the percolation problem on a 2-dimensional square lattice that reduces the number of degrees of freedom by 2 by removing every other point. Consider a unit square. Compute the probability that the points in the lower left and the upper right are connected by adding the probabilities of all relevant bond connections on the square. We can assign this as the new probability p' on a lattice in which the points in the upper left and lower right are removed. Then $p \to p'$ gives an approximate recursion formula. Find the prediction for p_c from this recursion formula and value of ν implied by Renormalization Group arguments.