Physics 212 – Final Exam

(due Friday, December 10)

I will use the quizzes and the final in this course to assign grades, keeping in mind that anyone who hands in all of the problem sets is assured a reasonable grade in the course. Because this exam will be graded, please abide by these rules:

- The quizzes are open-book. You may use any reference resources that you find. You may use your mathematical software such as MatLab or Mathematica. However, please do not collaborate with other students or ask help from other people—except that, if you have any questions about the quiz, please feel free to email me (mpeskin@slac.stanford.edu).
- The final is posted at the course web site:

  `https://www.slac.stanford.edu/~mpeskin/Physics212/`

  If there are corrections, the latest version can be found there.
- If you plan not to hand in the exam, this is acceptable, but please email me and let me know.
- Please write on your solution: "I acknowledge the Stanford Honor Code." and sign it.
- Please hand in the final to Gradescope by 4:00pm on Friday, Dec. 10. This is a sharp deadline. If this poses a problem for you, please email me in advance.

The final will be worth 50 points. Partial credit will be given.

1. The Potts model is a generalization of the Ising model, with a spin variable σ_i at each site i that takes the values $1, \cdots q$, where q is an integer. I will refer to σ as a "color" (red, green, etc.). The Hamiltonian is

$$\mathcal{H} = -K \sum_{i,\nu} \delta(\sigma_i, \sigma_{i+\nu}) \ , \tag{1}$$

where ν is an elementary lattice vector and $K > 0$. That is, the energy associated with a bond is lowered when the colors on adjacent sites are the same, and otherwise the bond energy is 0. The Potts model is related to a number of interesting statistical mechanics problems, include percolation, random resistor networks, and graph coloring. There are some reviews of the Potts model that you might find, but, fortunately, all of them are very difficult to read. So I hope that you can learn something about this model by working through this problem.

In this problem, the model is realized on a 2-dimensional, periodically connected square lattice with N sites.

(a) Show that, for the case $q = 2$, the Potts model is equivalent to the Ising model. Find J of the related Ising model in terms of the K of the Potts model.

(b) Set up the mean field theory of the Potts model. This is a little tricky. Since σ is a label, the object $\langle \sigma_i \rangle$ has no meaning. It is best to think about the probabilities with which the various colors appear. The disordered state will be the state in which all colors appear with the same probability. In an ordered state, there will be a dominant color, say, $\sigma = 1$, and the other colors will appear with equal but lower probability. Let the probability of $\sigma = 1$ be x, then the other colors will appear with probability

$$(1 - x)/(q - 1) \tag{2}$$

Show that the mean-field relation is

$$x = \frac{e^{4K\beta x}}{e^{4K\beta x} + (q-1)e^{4K\beta(1-x)/(q-1)}} \tag{3}$$

(c) Solve the equation (3) graphically for $q = 2$ and various values of β. Show that there is a second-order phase transition. Find T_c and show that it agrees with the prediction from the T_c of the 2-dimensional Ising model in mean field theory.

(d) Solve the equation (3) grapically for $q = 4$. Show that the phase transition is first-order (discontinuous). Find the transition temperature T_t (numerically) in this case.

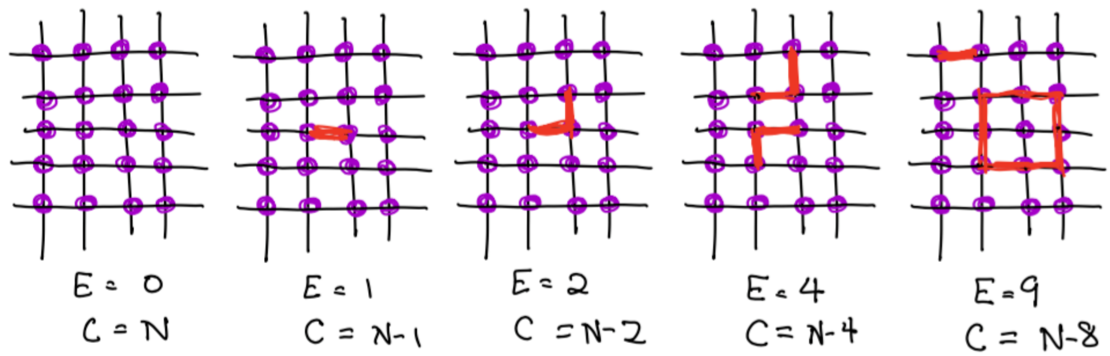
(e) What is the general situation? Where is the boundary between first- and second-order behavior?

You will see that mean-field theory predicts that the Potts model has a first-order phase transition for the cases $q = 3$ and $q = 4$. However, in these cases, exact analysis shows that the theory has a second-order (continuous) transition. However, for larger q, the transition is first-order. To understand this, let's look at other methods of solution.

(f) Work out the high-temperature expansion of the model for general q. Show that one can represent

$$e^{\beta K\delta(\sigma_i,\sigma_j)} = 1 + z\delta(\sigma_i, \sigma_j) \tag{4}$$

Find z. Find a graphical rule for working out the free energy of the Potts model as a power series expansion in z. Some representative diagrams are shown here:



(5)

Notice that every vertex on the lattice is used in each diagram. In the first diagram, each vertex is a separate disconnected component. In the second diagram, two vertices are linked by a bond into a single component. The last diagram has 9 edges and one closed loop or face (E = 9, F = 1) and still uses all N vertices (V= N).

(g) Work out the free energy of the model in powers of z, including terms through z^4.

(h) Here is a way to get a corresponding low-temperature expansion: First, go back to the high-temperature expansion. Note that the expression for Z is a finite polynomial in z. Each term is a diagram with $V = N$ vertices, E edges, F faces, and C connected components. Show that

$$Z = \sum_{diagrams} z^E q^C \ . \tag{6}$$

(i) Next, for each diagram, draw on the dual lattice a diagram dual to each diagram contributing to (6). To draw this diagram, draw an edge on the dual lattice for every edge for which the corresponding edge is not used on the original lattice. Thus, the first diagram in (5) has a dual diagram that includes every edge on the dual lattice. Physically, this is a diagram indicating an independent color at every point on the original lattice. Draw the dual diagrams for the other diagrams shown in (5).

(j) Relate the number of vertices, edges, faces, and connected components in the dual diagram $(\overline{V}, \overline{E}, \overline{F}, \overline{C})$ to those of the original diagram (V, E, F, C). In particular, $\overline{V} = V = N$ and $E + \overline{E} = 2N$. Using these relations, rewrite (6) in terms of the properties of the dual diagrams.

(k) Use Euler's topological relation

$$V - E + F = C \tag{7}$$

to put the result for Z in in (6) with a sum over the dual diagrams. There will be some prefactors that do not depend on the properties of the dual diagram.

(l) Show that this formula gives a low-temperature expansion for Z. Draw the diagrams that give the leading terms in the low-temperature limit.

(m) Assuming that there is only one phase transition in this model, use the duality to find the location of the phase transition.

(n) Go back to part (g) and use the series for the free energy to estimate the position of the singularity at which this series diverges. How does this point depend on q?

(o) Comparing the results of (m) and (n), conclude that the Potts model must have a first-order phase transition when q is very large.