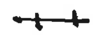
Physics 212 - Final Exam

Solutions

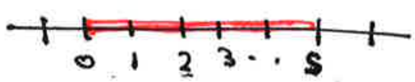
1.) a.) In 1-dimension, percolation occurs when there is a line of bonds that goes all of the way across the lattice

 = bond

 = no bond

The probability of this is p^N which $\to 0$ for $N \to \infty$ and any $p < 1$.

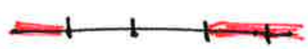
To count clusters of length s, you need to be more clever than I was on the exam. A configuration



has the weight $(1-p)\, p^s\, (1-p) = (1-p)^2 p^s$

However, there is danger of double-counting here. Let's include the $(1-p)$ on the left, but not on the right.

Also, there is the possibility of no cluster



which has probability $(1-p)^2$ but again omit the $(1-p)$ on the right.

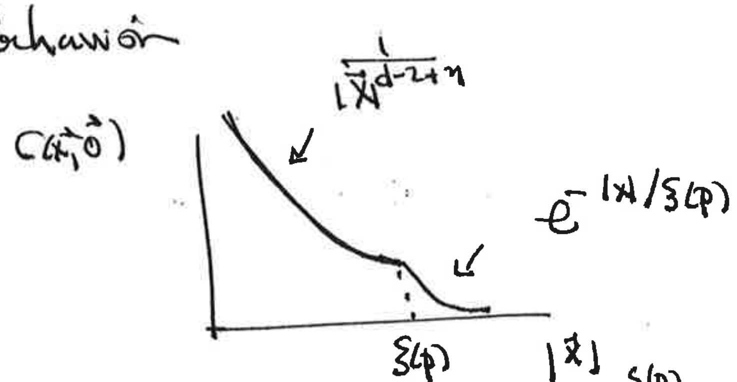
Then we count all possibilities for a given site as

$$\underbrace{(1-p)}_{\text{no cluster}} + \underbrace{\sum_{s=1}^{\infty} (1-p)\, p^s}_{\text{cluster of length } s}$$

$$= (1-p) + (1-p)\, p\, (1 + p + p^2 + \cdots)$$

$$= (1-p) + (1-p)\, p \frac{1}{1-p} = (1-p) + p = 1 \quad ✓$$

b.) Near $p = p_c$ $C(\vec{x}, \vec{0})$ has the almost scale invariant behavior



then $\displaystyle \int d^d x\; C(\vec{x}, 0) \sim \int_0^{\xi(p)} d^d x \frac{1}{|x|^{d-2+\eta}} \sim \frac{1}{\xi^{-2+\eta}} \sim \xi(p)^{2-\eta}$

then $N(p) \sim \xi^{2-\eta} \sim (p_c - p)^{-(2-\eta)\nu}$

or $N(p) \sim (p_c - p)^{-\gamma}$ $\quad \gamma = \nu(2-\eta)$

c.) If s is the number of bonds in a cluster and $n(s)$ is the number density of clusters of size s, we can estimate $\langle s \rangle$ by

$$\langle s \rangle = \frac{\sum_s s^2 n(s)}{\sum_s s\, n(s)}$$

As $p \to p_c$ from below, the denominator stays fixed as numerator diverges. Then the divergence in $\langle s \rangle$ comes from the numerator

$$N(p) \sim \int ds\ s^2\, n(s) .$$

In a scale-invariant description $n(s) \sim s^{-1/\sigma}$ until $s \sim \xi^d$. Then

$$N(p) \sim (\xi^d)^{(3-1/\sigma)} \sim (p_c - p)^{-\nu d(3-1/\sigma)}$$

equating this to $(p_c-p)^{-\gamma}$

$$\gamma = \nu(2-\eta) = \nu d(3-1/\sigma)$$

$$(2-\eta) = 3d - d/\sigma$$

$$d/\sigma = 3d - 2 + \eta$$

$$\sigma = \frac{d}{3d-2+\eta}$$

In studies of percolation in different dimensions (D. Stauffer, Phys. Repts. 54, 1 (1979))

d	γ	ν	σ
2	2.43	1.3	0.39
3	1.7	0.8	0.4

d.) On a square lattice of N sites, there are $2N$ bonds. The number of placements of bonds with 1 bond is $2N$. The number of placements of 2 bonds is

$$\frac{2N(2N-1)}{2!}$$

Similarly, the denominator is

$$1 + 2Nq + \frac{2N(2N-1)}{2!} q^2 + \frac{2N(2N-1)(2N-2)}{3!} q^3 + \cdots$$

$$= (1+q)^{2N}$$

e.) Now compute the numerator of $N(q) = \sum_x C(\vec{x}, 0)$

Let's count the q and q^2 terms first. In general, let s be the number of bonds,

For $s=1$, there are 2 shapes 

Actually these are related by rotations.

For each $\vec{0}$ can be located in each of $n = 2$ places, where $\underline{n}$ is the number of $\underline{\text{sites}}$ in the cluster.
Also $\sum_x$ gives $n = 2$. Then $\#$ of $\vec{0}$ positions $\times \sum_x = n^2$

so

	$\underline{\#}$	$\#$ of $\vec{0}$ pos $\times \sum_x = n^2$
	4	4

Given that this is the cluster connected to $\vec{0}$, we can put down additional bonds as long as these are disconnected from the cluster, that is, bonds that do not touch the cluster. 7 bonds are excluded

so finally, for $\underline{s = 1}$

	$\underline{\#}$	$\underline{n^2}$	$\underline{q^s}$	$\underline{\text{disconnected}}$
	2	4	q	$\left(1 + (2N-7)q + \frac{(2N-7)(2N-8)}{2!} q^2 + \cdots\right)$

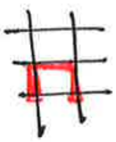
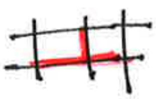
For $\underline{s = 2}$

	$\underline{\#}$	$\underline{n^2}$	q^s	disconnected
	2	9	q^2	$(1 + (2N-10)q + \cdots)$
	4	9	q^2	$(1 + (2N-10)q + \cdots)$

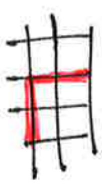
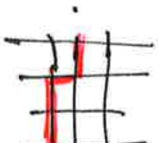
$\Uparrow$

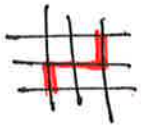
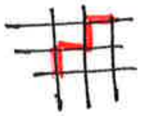
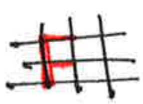
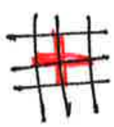
Starting with $n = 3$ I will count rotations and reflections here.

$S = 3$

	#	n^2	q^s	disconnected
	2	16	q^3	$(1 + (2N-13)q + \cdots)$
	8	16	q^3	$(1 + (2N-13)q + \cdots)$
	4	16	q^3	$(1 + (2N-13)q + \cdots)$
	4	16	q^3	$(1 + (2N-12)q + \cdots)$
	4	16	q^3	$(1 + (2N-13)q + \cdots)$

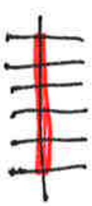
$S = 4$

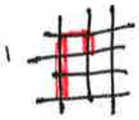
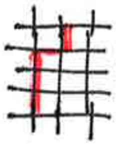
	#	n^2	q^s	disconnected
	2	25	q^4	$(1 + (2N-16)q + \cdots)$
	8	25	q^4	$(1 + (2N-16)q + \cdots)$
	4	25	q^4	$(1 + (2N-16)q + \cdots)$
	8	25	q^4	$(1 + (2N-15)q + \cdots)$
	8	25	q^4	$(1 + (2N-16)q + \cdots)$

	4	25	q^4	$(1+(2N-16)q+\cdots)$
	4	25	q^4	$(1+(2N-16)q+\cdots)$
	8	25	q^4	$(1+(2N-16)q+\cdots)$
	8	25	q^4	$(1+(2N-15)q+\cdots)$ -
	8	25	q^4	$(1+(2N-16)q+\cdots)$
	1	16	q^4	$(1+(2N-12)q+\cdots)$ ←
	4	25	q^4	$(1+(2N-16)q+\cdots)$
	8	25	q^4	$(1+(2N-16)q+\cdots)$
	8	25	q^4	$(1+(2N-15)q+\cdots)$ -
	1	25	q^4	$(1+(2N-16)q+\cdots)$

finally, <u>$s=5$</u>

not needed up to q^5

	2	36	q^5	$(1+\cancel{(2N-19)q}+\cdots)$
	8	36	q^5	

	8	36
	8	36
	8	36
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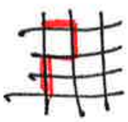
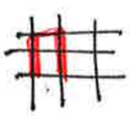
	8	36
	4	36
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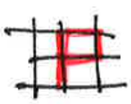	8	25 ←
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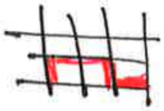
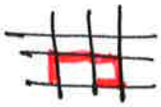
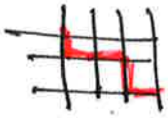
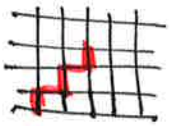
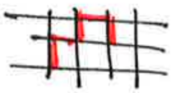
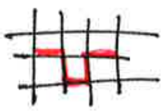
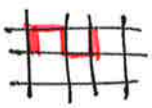
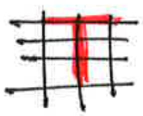
	8	36
	8	36
	8	36
	4	36
	8	36
	4	36
	48	36
	4	36
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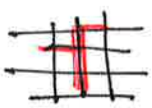
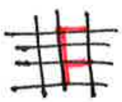
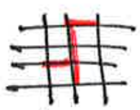
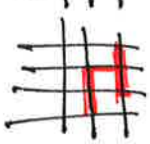
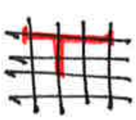
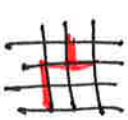
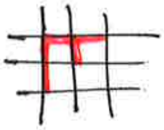
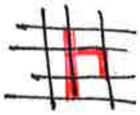
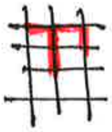
	8	36
	8	36
	8	36
	8	36
	8	36
	8	36

	8	36
	8	36
	8	36
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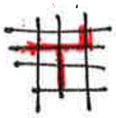	8	36
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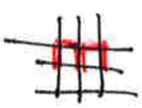
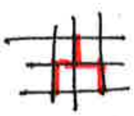
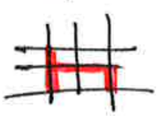
	4	36
	4	36
	8	36
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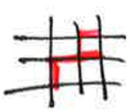	8	36
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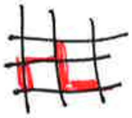
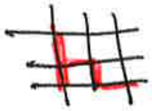
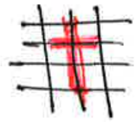
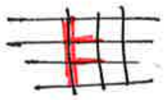
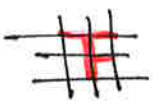
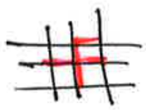
 8 36

 8 36

 4 36

 4 36

 8 36

 8 36

In all, we have for the numerator:

$$q \cdot 2 \cdot 4 \cdot (1+q)^{2N-7}$$

$$+ \; q^2 \cdot 6 \cdot 9 \cdot (1+q)^{2N-10}$$

$$+ \; q^3 \left\{ 18 \cdot 16 \cdot (1+q)^{2N-13} + 4 \cdot 16 \cdot (1+q)^{2N-12} \right\}$$

$$+ \; q^4 \left\{ 59 \cdot 25 \cdot (1+q)^{2N-16} + 24 \cdot 25 \cdot (1+q)^{2N-15} + 1 \cdot 16 \cdot (1+q)^{2N-12} \right\}$$

$$+ \; q^5 \left\{ 344 \cdot 36 + 8 \cdot 25 \right\} + \cdots$$

divide by $(1+q)^{2N} \longrightarrow$

then

$$N(q) = 8q(1-7q+28q^2-84q^3+210q^4)$$
$$+ 54q^2(1-10q+55q^2-220q^3)$$
$$+ q^3(288(1-13q+91q^2)$$
$$+ 64(1-12q+78q^2))$$
$$+ q^4(1475(1-16q)+600(1-15q)$$
$$+ 16(1-12q))$$
$$+ q^5\{12584\} + O(q^6)$$
$$= 8q - 2q^2 + 36q^3 - 123q^4 + 792q^5$$
$$+ \cdots$$

f.) $$(1-x/a)^{-\gamma} = 1 + \gamma\frac{x}{a} + \frac{\gamma(\gamma+1)}{2!}\left(\frac{x}{a}\right)^2$$
$$+ \cdots + \frac{\gamma(\gamma+1)\cdots(\gamma+n-1)}{n!}\left(\frac{x}{a}\right)^n + \cdots$$

ratio of coefficients $$\frac{a_n}{a_{n-1}} = \frac{\gamma+n-1}{n}\,\frac{1}{a}$$

then for large n

$$\frac{a_n}{a_{n-1}} = \frac{1}{a}\left(1 + \frac{\gamma - 1}{n} + O\left(\frac{1}{n^2}\right)\right)$$

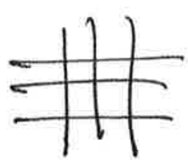
The leading singularity is then estimated by

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \frac{1}{a}$$

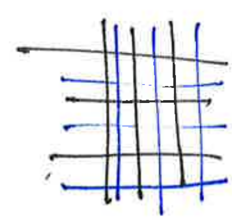
Unfortunately, this works well when the coefficients are all positive. For this series, we find negative coefficients, indicating a leading singularity on the negative real z axis. This should have been no surprise, since there is a singularity at $z = -1$.

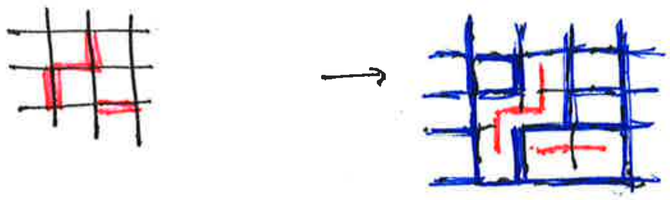
So, apologies, we cannot estimate z_c by so simple a method.

9.) Draw an array of bonds on the square lattice:



This pattern gives a pattern of bonds on the dual lattice: If a bond is colored on the original lattice, the dual bond is uncolored, and vice versa.





For small p, the bonds on the original lattice are mostly uncolored, and the bonds on the dual lattice are mostly colored. So for small p, we have $p < p_c$ on the original lattice and $p > p_c$ on the dual lattice.

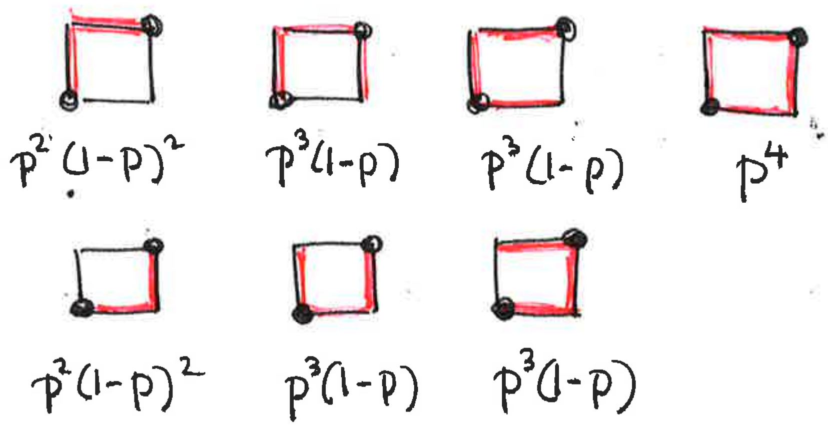
The bond placement probabilities on the dual lattice are the same as those on the original lattice with the replacement $p \rightarrow (1-p)$

So, if there is one phase transition in this problem, it must be at

$$p = (1-p) = \frac{1}{2}$$

This does turn out to be the exact location of p_c.

h.) On the original lattice, the lower left and upper right corners of a square are connected by the following placements of bonds



The sum of the weights is

$$p^2\left[(1-p)^2 + 2p(1-p) + p^2\right]$$

$$+ \quad p^2\left[(1-p)^2 + 2p(1-p)\right]$$

$$= \quad 2p^2 - p^4$$

so the recursion formula is

$$p' = 2p^2 - p^4$$

Notice that p decreases at small p and increases as $p \to 1$, as expected.

There is a fixed point at

$$p_c = 2p_c^2 - p_c^4$$

The solution is $\quad p_c = 0.61803$

Near the fixed point the evolution is $p_c + \Delta p \to p_c + \Delta p'$

$$\Delta p' = (4p_c - 4p_c^3)\,\Delta p$$

$$= 1.528\ \Delta p = \lambda\ \Delta p$$

After n iterations, the lattice spacing is changed by $(\sqrt{2})^n$ and $\Delta p^{(n)} = \lambda^n \Delta p$. When $\Delta p^{(n)}$ is of order 1, the lattice spacing is

$$\xi \sim 2^{n/2} \quad \text{with} \quad n = \frac{(-\log(p_c - p))}{\log \lambda}$$

$$= \exp\left[-\log(p_c - p)\ \frac{\log 2}{2\log\lambda}\right]$$

$$= (p_c - p)^{-\left(\frac{\log 2}{2\log\lambda}\right)}$$

so

$$\nu = \frac{\log 2}{2\log\lambda} = 0.818$$