Physics 212 - Final Exam
Solutions

1.) a.) The partition function of the Potts model for $q=2$ is

$$Z_P = \sum_{\sigma_i=1,2} e^{\beta K \sum_{ir} \delta(\sigma_i, \sigma_{i+r})}$$

Compare this to the Ising model

$$Z_{Is} = \sum_{S_i=\pm 1} e^{\beta J \sum_{ir} S_i S_j}$$

For each bond

$\sigma_i = \sigma_j$	$e^{\beta K}$	$S_i = S_j$	$e^{+\beta J}$
$\sigma_i \neq \sigma_j$	1	$S_i \neq S_j$	$e^{-\beta J}$

so

$$Z_P = e^{\beta \frac{K}{2} \cdot 2N} \sum_{\sigma_i} \prod_{\text{bonds}} \left[e^{\frac{\beta K}{2}} \delta(\sigma_i, \sigma_j) + e^{-\frac{\beta K}{2}} \delta(\sigma_i \neq \sigma_j) \right]$$

so $$Z_P = e^{\beta K N} \cdot Z_I \left(J = \frac{K}{2} \right)$$

b.) Let $\sigma = 1$ be the dominant color, and let all other colors have equal probability. There will be symmetrical solutions with $\sigma = 2, 3, \ldots$ as the dominant color. Let the probability of finding $\sigma_i = 1$ be x. In mean field theory, we consider this probability to be independent at every site. Then the probability of finding $\sigma_i = 2$ is

$$y = \left(\frac{1-x}{q-1}\right)$$

so that

$$\sum_{\sigma = 1 \ldots q} P(\sigma_i) = P(\sigma_i = 1) + \sum_{\sigma_i > 1} P(\sigma_i \neq 1)$$

$$= x + (q-1) \cdot \frac{(1-x)}{(q-1)} = 1 \checkmark$$

Now evaluate x in this approximation. The statistical weight for $\sigma_i = 1$ on the bond (i,j)

$$e^{\beta K \langle \delta(1, \sigma_j) \rangle} = e^{\beta K x}$$

The statistical weight for $\sigma_i = 2$ (or any other color) is

$$e^{\beta K \langle \delta(2, \sigma_j) \rangle} = e^{\beta K y}$$

Then the probability to find $\sigma_i = 1$ is

$$x = \frac{e^{2d\beta K x}}{e^{2d\beta K x} + (q-1)\, e^{2d\beta K y}}$$

For a 2-d lattice, the number of bonds at a site is $2d = 4$, so

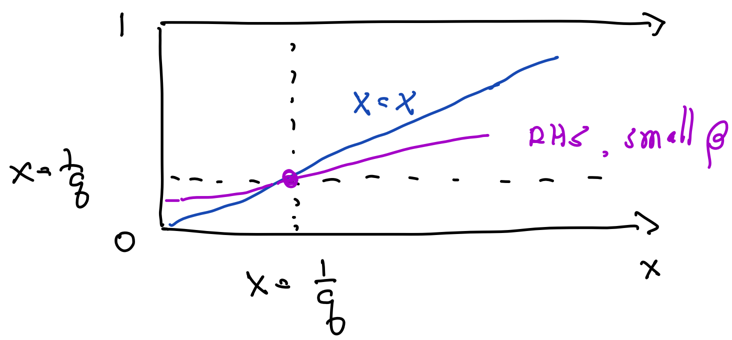
$$x = \frac{e^{4\beta K x}}{e^{4\beta K x} + (q-1)\, e^{4\beta K (1-x)/(q-1)}}$$

Similarly

$$y = \frac{e^{4\beta K (1-x)/(q-1)}}{e^{4\beta K x} + (q-1)\, e^{4\beta K (1-x)/(q-1)}}$$

So $$x + (q-1)\, y = 1$$ as required

c.) The RHS of the equation for x has the form for small β



That is
$$\begin{aligned} \text{RHS} &\to 0 && \text{for } x \to -\infty \\ &= \tfrac{1}{q} && \text{for } x = \tfrac{1}{q} \\ &\to 1 && \text{for } x \to \infty \end{aligned}$$

Let $\xi = (x - 1/q)$ then $\dfrac{1-x}{q-1} = \dfrac{1}{q} - \dfrac{1}{q-1}\xi$

$$e^{4\beta K x} = e^{4\beta K/q} \left(1 + 4\beta K \xi + \tfrac{1}{2}(4\beta K)^2 \xi^2 + \cdots\right)$$

$$e^{4\beta K \frac{1-x}{q-1}} = e^{4\beta K/q}\left(1 - \frac{4\beta K}{q-1}\xi + \frac{1}{2}\frac{(4\beta K)^2}{(q-1)^2}\xi^2 + \cdots\right)$$

$$e^{4\beta K x} + (q-1)\, e^{4\beta K \frac{1-x}{q-1}}$$

$$= q + (4\beta K - 4\beta K)\xi + \tfrac{1}{2}(4\beta K)^2\left(1 + \frac{1}{q-1}\right)\xi^2$$

$$= q\left(1 + \tfrac{1}{2}(4\beta K)^2 \frac{1}{q-1}\xi^2 + \cdots\right)$$

then the self-consistency equation is

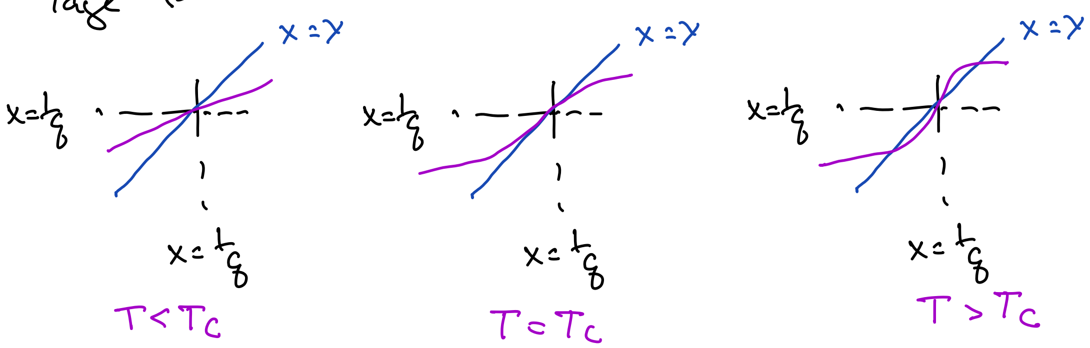
$$\frac{1}{q} + \xi = \frac{1}{q}\,\frac{\left(1 + 4\beta K\xi + \frac{1}{2}(4\beta K)^2\xi^2 + \cdots\right)}{\left(1 + \frac{1}{2}(4\beta K)^2\xi^2/(q-1) + \cdots\right)}$$

$$= \frac{1}{q}\left(1 + 4\beta K\xi + \tfrac{1}{2}(4\beta K)^2\left(1 - \frac{1}{q-1}\right)\xi^2 + \cdots\right)$$

For $q = 2$, the slopes match at $\xi = 0$ when

$$4\beta K/2 = 1$$

The ξ^2 term is 0, and you can check that the ξ^3 term is negative. So the evolution from small to large β is



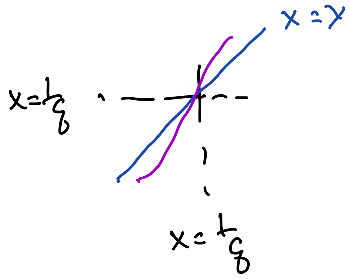
This is a continuous or second-order transition, as in the mean field theory of the Ising model.

The condition for T_c is

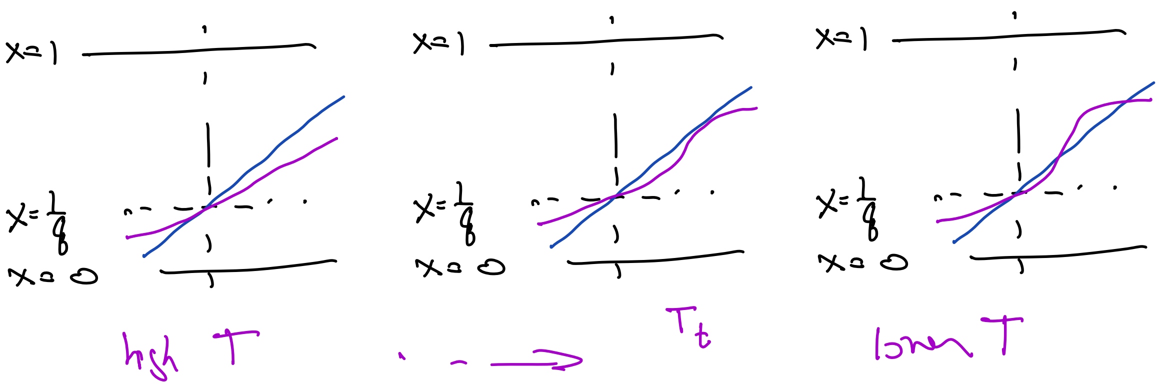
$$T_c = \frac{4K}{2} = 4J$$

exactly as for the 2-d Ising model in mean field theory.

d) For $q > 2$, the ξ^2 term in the self-consistency equation is <u>positive</u>. So when the slopes match, the curves join like



and there already must have been a solution at a higher temperature. By graphing the RHS (or, better, graphing RHS(x) − x vs. x you can see that the evolution is

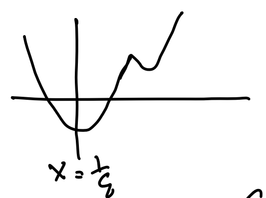


For $q = 4$, the RHS touches the $x = x$ line at $4\beta K \approx 3.2$

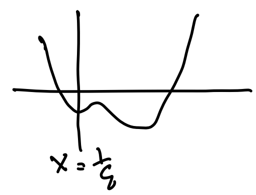
or $T_t = 1.25\,K$, which is greater than

$T_c = \frac{4K}{q} = 1.K$ where the slopes match at $x = \frac{1}{q}$

Just below T_t, the free energy as a function of x looks like



Near T_c, the free energy as a function of x looks like



Somewhere between T_t and T_c, there is a discontinuous (first order) transition.

e) The same logic applies for any $q > 2$. In mean field theory, there is a first-order transition. However, from high temperature expansions and exact results, the Potts model in 2-d has a first-order transition only for $q > 4$.

f) The Potts model partition function can be written

$$Z_P = \sum_{\sigma_i} \prod_{\text{bonds}} \left[1 + z\,\delta(\sigma_i, \sigma_j)\right]$$

where if $\sigma_i = \sigma_j$ $\quad [\;] = e^{\beta K}$

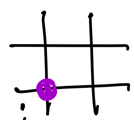
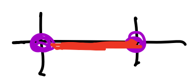
$\sigma_i \neq \sigma_j$ $\quad [\;] = 1$

so

$$z = (e^{\beta K} - 1)$$

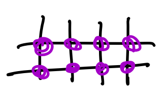
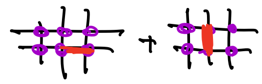
At high T, $z \approx \beta K = K/T \ll 1$. At low T, $z \approx e^{\beta K} \gg 1$.

To represent the power series expansion in z, color a bond on the lattice when using the z term, and leave the bond uncolored when using the 1 term. Vertices joined by a colored bond must have the same color. So

$\sum_{\sigma_i}$  $= q \qquad \sum_{\sigma_i \sigma_j}$  $= q$

etc. Then

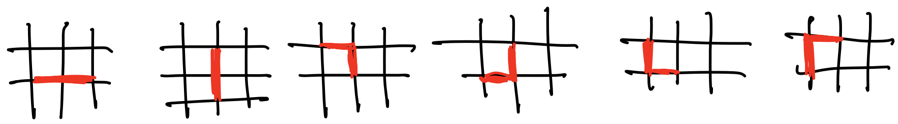
$$Z = q^N + 2Nzq^{N-1} + \cdots$$

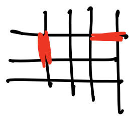
 

g.) At higher orders in z, we must count bonds more carefully. However, there is a check. The total number of configurations with m colored bonds is

$$\frac{2N(2N-1)(2N-2)\cdots(2N-m+1)}{m!}$$

z^2 total $= \frac{2N(2N-1)}{2}$

 $6N \cdot q^{N-2}$

 $\left(\frac{2N(2N-1)}{2} - 7N\right) \cdot q^{N-2}$

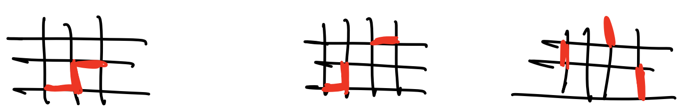
$= N(2N-7)\, q^{N-2}$

sum: $(2N^2 - N)\, q^{N-2} = \frac{2N(2N-1)}{2}\, q^{N-2}$ ✓

Actually, we didn't have to work this hard. All of the diagrams have weight q^{N-2}, so all we need is the sum.

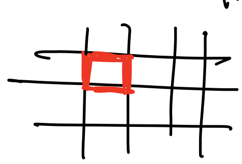
z^3: total $= \frac{2N(2N-1)(2N-2)}{3 \cdot 2}$

the three types of diagram are



and all have weight q^{N-3}

z^4: Almost all diagrams have weight q^{N-4}. There is one class that is different

 $q^{N-3} \cdot N$

so

$$Z = q^N \Big\{ 1 + 2Nq^{-1}z + \Big[\frac{(2N)^2}{2} - N\Big] z^2 q^{-2}$$

$$+ \Big[\frac{(2N)^3 - 12N^2 + 4N}{6}\Big] z^3 q^{-3}$$

$$+ \Big[\frac{(2N)^4 - 48N^3 + 44N^2 - 12N}{24} - N\Big] z^4 q^{-4}$$

$$+ N \cdot z^4 q^{-3} + \cdots \Big\}$$

$$= q^N \Big\{ 1 + 2N\Big(\frac{z}{q}\Big) + \Big[\frac{(2N)^2}{2} - N\Big]\Big(\frac{z}{q}\Big)^2$$

$$+ \Big[\frac{(2N)^3}{3!} - 2N^2 + \frac{2}{3}N\Big]\Big(\frac{z}{q}\Big)^3$$

$$+ \Big[\frac{(2N)^4}{4!} - 2N^3 + \frac{11}{6}N^2 - \frac{3}{2}N + qN\Big]\Big(\frac{z}{q}\Big)^4$$

$$+ \cdots \Big\}$$

$$= q^N \exp\Big[2N\Big(\frac{z}{q}\Big) - N\Big(\frac{z}{q}\Big)^2 + \frac{2}{3}N\Big(\frac{z}{q}\Big)^3$$

$$+ (q - 3/2)\, N\Big(\frac{z}{q}\Big)^4 + \cdots \Big]$$

$$= e^{-\beta F}, \text{ so}$$

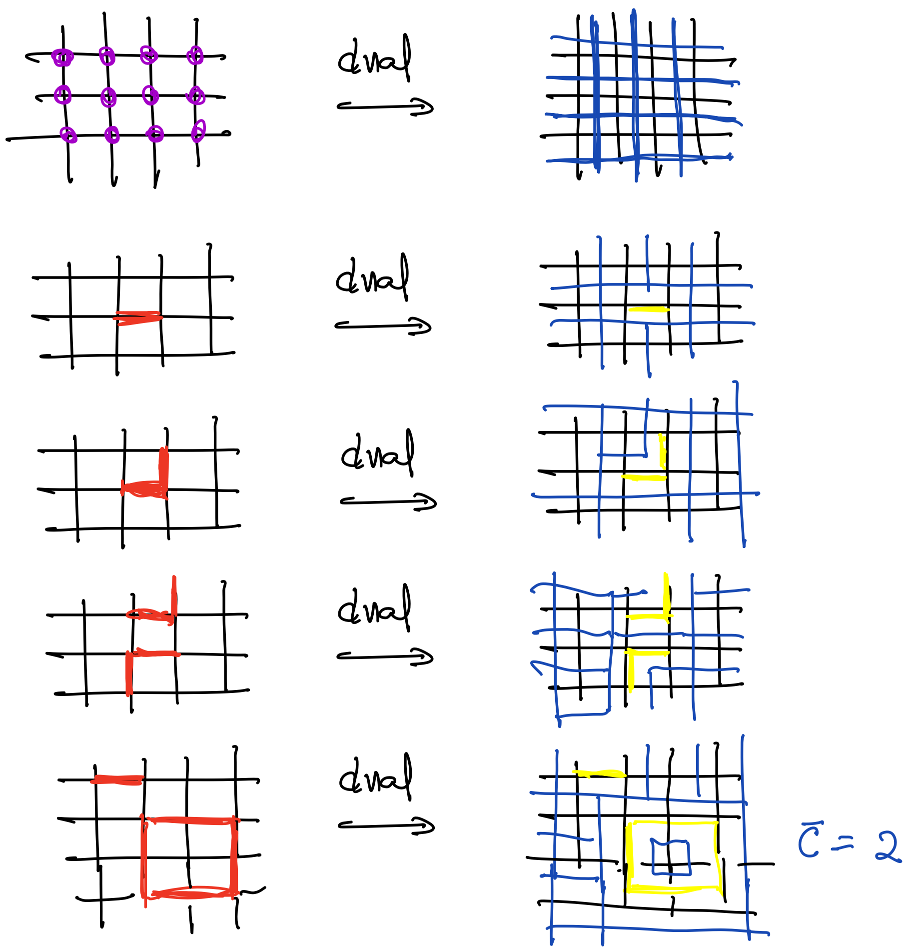
$$F = -\frac{N}{\beta}\Big\{ \log q + 2\frac{z}{q} - \Big(\frac{z}{q}\Big)^2 + \frac{2}{3}\Big(\frac{z}{q}\Big)^3$$

$$+ (q - 3/2)\, N(z/q)^4 + \cdots \Big\}$$

h.) The general counting is that each edge gives a factor of z and each connected component, e.g. or , gives 1 factor of q.

So

$$Z = \sum_{\text{diagrams}} z^E q^C$$

i.) 

j.) Each diagram uses all vertices on its own lattice, so $\bar{V} = V = N$

Each edge is used by either the original or the dual diagram

$$E + \bar{E} = 2N$$

Faces and connected components are interchanged

$$\bar{F} = C \qquad \bar{C} = F$$

Then

$$Z = \sum_{\substack{\text{dual} \\ \text{diagrams}}}' z^{2N-\bar{E}} q^{\bar{F}}$$

k.) $\bar{F} = \bar{C} + \bar{E} - \bar{V} = \bar{C} + \bar{E} - N$

$$Z = \sum_{\substack{\text{dual} \\ \text{diagrams}}}' z^{2N-E} q^{\bar{C}+\bar{E}-N}$$

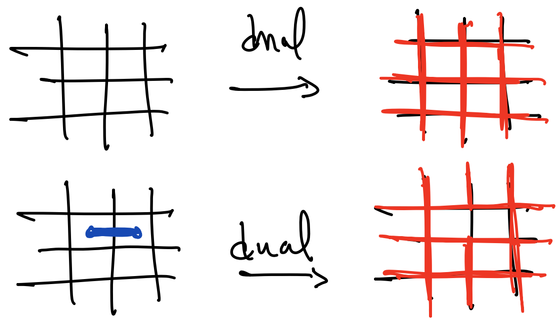
$$= z^{2N} q^{-N} \sum_{\substack{\text{dual} \\ \text{diagrams.}}}' \left(\frac{q}{z}\right)^{\bar{E}} q^{\bar{C}}$$

So

$$Z_{\text{Potts}}(z,q) = (z^2/q)^N \, Z_{\text{Potts}}\left(\frac{q}{z}, q\right)$$

l.) On the right, we have the same diagram series except that now the expansion parameter is q/z

This is a low-temperature series, valid when $z \to \infty$, $T \to 0$. The leading diagrams are



etc.

m.) If there is 1 singularity in Z_{Potts}, it must occur where

$$z_* = \frac{q}{z_*} \Rightarrow z_* = \sqrt{q}$$

The position, in terms of K, is

$$(e^{\beta K} - 1) = \frac{q}{e^{\beta K} - 1}$$

$$(e^{\beta K} - 1) = \sqrt{q}$$

$$\beta K = \log(\sqrt{q} + 1)$$

For the Ising case $q = 2$, $K = 2J$

and we find $T_c = 2J / \log(\sqrt{2} + 1)$!

n.) Except for a few diagrams at each order, the high-temperature series is a series in (z/q). So I expect that this series will converge out to $z \sim q^1$. By duality, the low-temperature series will be valid for $z > O(1)$.

However, the phase transition is at $z_* \sim q^{1/2}$. So at z_*, there are two distinct thermodynamic states that crossover in free energy. This is a first-order transition.