June 1

Critical Exponents in 2 Dimensions

In the previous lecture, I discussed the phenomenon of *universality* in critical phenomena in 3 dimensions. I gave an explanation in terms of Renormalization Group (RG) flows to a unique fixed point. We analyzed the RG flow of Landau theory and identified this fixed point as the Wilson-Fisher fixed point that is found in weak coupling for dimensions close to $d = 4$.

In that discussion, I noted that the number of relevant operators at the Gaussian fixed point increases as the dimension d is reduced below $d = 3$. For an interaction

$$C_p M^{2p}$$

the scaling at the Gaussian fixed point is

$$C_p \rightarrow C_p' = \lambda^{2p-(p-1)d} C_p$$

As $d \to 2$, all interactions $C_p M^{2p}$ become relevant. This ought to be a sign that there exist a large number of nontrivial fixed points—and different sets of critical exponents visible in experiment—in 2-dimensional systems.

In this lecture, I will present an infinite family of distinct critical points in 2 dimensions. We will discover these critical points using a powerful method for analyzing scale-invariant models, an extension of scale symmetry called *conformal invariance*.

To define conformal invariance, look back at the scale invariance of the Gaussian model

$$\mathcal{H} = \int d^d x \; \frac{1}{2} (\vec{\nabla} s)^2$$

The expression is invariant under

$$S(x) \rightarrow \frac{1}{\lambda^A} S(x/\lambda)$$

where

$$A = \frac{d-2}{2}$$

To prove this, let

$$x' = x/\lambda$$

Then

$$\vec{\nabla} s \rightarrow \vec{\nabla} \frac{1}{\lambda^{d/2-1}} s(x/\lambda) = \frac{1}{\lambda^{d/2}} (\vec{\nabla} s)(x/\lambda)$$

and

$$d^d x = d^d x' \, \lambda^d$$

In fact, this model is invariant under a larger group of symmetries. Write the scale transformation in its infinitesimal form as

$$\vec{x} \rightarrow \vec{x} - \epsilon \vec{x}$$

We might generalize this to a transformation

$$\vec{x} \to \vec{x}' = \vec{x} - \epsilon \vec{v}(x)$$

Then

$$d^d x = d^d x' \left[\det\left(1 - \epsilon \frac{\partial v^i}{\partial x^j}\right)\right]^{-1} = d^d x' \left(1 + \epsilon \vec{\nabla}\cdot\vec{v}\right)$$

If we let

$$S(x) \to \left(1 - \epsilon \frac{d-2}{2d} \vec{\nabla}\cdot\vec{v}\right) S(x - \epsilon v)$$

then

$$\nabla^i S \to \nabla^i S(x) - \epsilon \frac{d-2}{2d} \vec{\nabla}\cdot\vec{v}\ \nabla^i S(x') - \epsilon (\nabla^i v^j) \nabla^j S(x')$$
$$- \epsilon \frac{d-2}{2d} \nabla^i (\vec{\nabla}\cdot\vec{v}) S(x')$$

The Hamiltonian of the Gaussian model then transforms to

$$\mathcal{H} \to \int d^d x' \Big\{ \tfrac{1}{2}(\vec{\nabla} S)^2(x') + \tfrac{1}{2} \epsilon (\vec{\nabla}\cdot\vec{v}) (\nabla S)^2$$
$$- \tfrac{1}{2}\epsilon \frac{d-2}{d} (\vec{\nabla}\cdot\vec{v}) (\vec{\nabla} S)^2 - \tfrac{1}{2} 2\epsilon (\nabla^i v^j) \nabla^j S \nabla^i S$$
$$- \tfrac{1}{2}\epsilon \cdot \frac{d-2}{d} \nabla^i (\vec{\nabla}\cdot\vec{v})\ S(x') \nabla^i S(x')$$

If $\nabla^i(\vec{\nabla}\cdot\vec{v})$ is a *constant*, the last term is a total divergence

$$\int d^d x' \; \nabla^i S^2(x')$$

and integrates to zero. Then $H[S]$ transforms to

$$H(s) \to \int d^d x' \; \frac{1}{2}\left\{ (\vec{\nabla} S)^2 - \epsilon \left(\nabla^i v^j + \nabla^j v^i - \frac{2\delta^{ij}}{d} \vec{\nabla}\cdot\vec{v} \right) \nabla^i S \nabla^j S \right\}$$

Any $v^i(x)$ gives a symmetry of H if

$$\nabla^i v^j + \nabla^j v^i = \frac{2}{d} \vec{\nabla}\cdot\vec{v}$$

and $(\vec{\nabla} \cdot \vec{v})$ is at most linear in x. The general form allowed by these conditions is

$$v^i = \eta x^i + a^i x^2 + 2 \vec{a}\cdot\vec{x} \, x^i$$

This adds $(1 + d)$ parameters to the usual symmetries of translation and rotation invariance. The resulting group is called the *conformal group.* It has

$$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)(d+2)}{2}$$

parameters. The conformal group is isomorphic to $SO(d+1,1)$. The transformations generated by the a^i are called *special conformal tranformations.*

The simple Hamiltonian of the Gaussian model turns out not to be exceptional in possessing these extra symmetries. Almost always, a *scale invariant* theory is also *conformally invariant.* A conformal transformation is locally a scale transformation in which the scale factor is a function of x. The criterion for local scale invariance is the vanishing of the trace of the stress tensor,

$$T_{ii} = 0$$

The criterion for global scale invariance is slightly weaker, and only in exceptional cases does it make a difference. Polchinski (Nucl. Phys. B303, 226 (1988)) has given a detailed analysis of the restricted circumstances in which scale invariance does not imply conformal invariance.

In the 1970's, *Polyakov* tried to use the extra constraints of conformal invariance to compute critical exponents. This program led to some interesting analysis but did not succeed. However, we will see that conformal invariance is a much more powerful constraint in 2 dimensions. In $d = 2$, then, significant results can be obtained. The central reference for this theory is Belavin, Polyakov, and Zamolodchikov, Nucl. Phys. B241, 333 (1984).

Looking back at the argument for the conformal symmetry of the Gaussian model, we see that, if $d = 2$, there is no requirement that $(\vec{\nabla} \cdot vv)$ should be linear in x. The conformal symmetry group is then much larger in $d = 2$. It is useful to adopt complex coordinates,

$$z = \frac{x^1 + i x^2}{\sqrt{2}} \qquad \bar{z} = \frac{x^1 - i x^2}{\sqrt{2}}$$

Then

$$\int d^2x \; \frac{1}{2}(\vec{\nabla} S)^2 = \int dz\, d\bar{z} \; \partial_z S \; \partial_{\bar{z}} S$$

Any transformation

$$S(z, \bar{z}) \rightarrow S(\, f(z), \bar{f}(\bar{z}))$$

where $f(z)$ is an analytic function is a symmetry of the model. It is well known that analytic functions are *conformal mappings* in the sense used in complex analysis. That is, these functions are mappings of the complex plane that are local scale transformations and preserve angles.

There is a special set of conformal transformations in 2 dimensions that map the complex plane into itself by a 1-to-1 mapping. This is the group of *projective* or *fractional linear transformations*,

$$z \rightarrow \frac{az+b}{cz+d}$$

canonically normalized such that

$$ad - bc = 1$$

The infinitesimal form of this transformation is

$$z \rightarrow \alpha + \beta z + \gamma z^2$$

These are the conformal transformations, including translations and rotations, that generalize to higher dimensions. With independent parameters for the transformation of z and $\overline{z}$ (or, equivalently, treating α, β, γ as complex parameters), these transformations have 6 degrees of freedom. The group of projective transformations is isomorphic to $SO(3,1) = SL(2,C)$.

The more general conformal transformations shift z by any positive or negative power of z. These transformations are generated by infinitesimal motions of the form

$$f(z) \rightarrow (1+\epsilon L_n) f(z) \qquad \text{with} \qquad L_n = z^{n+1} \frac{d}{dz}$$

The L_n satisfy the algebra

$$[L_n, L_m] = (n-m) L_{n+m}$$

The operator L_0 is a pure scale transformation of z.

It turns out, though, that this equation does not quite give the correct commutation relation of scale transformations. Scale transformations involve the subtlety that they are generated by an operator, the energy-momentum tensor, which has a singularity in its 2-point correlation function strong enough to affect the commutator of these operators. These operator and Hilbert space considerations apply even in classical statistical mechanics. The relevant Hilbert space is that one on which the transfer matrix operates.

In any event, careful computation of the commutator of scale transformations gives the modified commutation relation with an extra c-number term on the right-hand side.

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\,\delta_{n+m,0}$$

This is called the *Virasoro algebra.* The extra term, called a *central charge*, is normalized so that the Gaussian model, the scale-invariant model of a single free boson field, has $c = 1$. In general $c > 0$. The linear term in n can be adjusted by adding a constant to L_0; the term proportional to n^3 is intrinsic. One canonically chooses the structure above, for which there is no central charge in the commutators of the projective transformations generate by L_{-1}, L_0, L_1.

In any Lie algebra, the generators much satisfy the *Jacobi identity*,

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

To test the consistency of the Virasoro algebra, represent the central charge by $C(n)$, which must obey $C(-n) = C(n)$. Compute the Jacobi identity for L_n, L_{-m}, L_{-p}, with $m, p, n > 0$ and $n = m + p$. We find

$$
\begin{aligned}
&[L_n,[L_{-m},L_{-p}]] + [L_{-m},[L_{-p},L_n]] + [L_{-p},[L_n,L_{-m}]] \\
&= [L_n\ (p-m)L_{-n}] + [L_{-m},(-p-n)L_m] + [L_{-p},(n+m)L_p] \\
&= 2n(p-m)L_0 + 2m(p+n)L_0 - 2p(n+m)L_0 \\
&\quad + C(n)(p-m) + C(m)(p+n) - C(p)(n+m)
\end{aligned}
$$

In the result, the first line completely cancels. The second line simplifies to

$$
C(n)(p-m) - n\left(C(p) - C(m)\right) + \left(p\,C(m) - m\,C(p)\right)
$$

with $n = (m+p)$. This equation is satisfied for $C(n) = n$. For $C(n) = n^3$,

$$
\begin{aligned}
&(p+m)^3(p-m) - (p+m)(p^3-m^3) + pm^3 - mp^3 \\
&= (p-m)\left[(p+m)^3 - (p+m)(p^2+mp+m^2) - mp(p+m)\right] \\
&= (p-m)(p+m)\left[(p^2+2pm+m^2) - (p^2+mp+m^2) - mp\right] = 0
\end{aligned}
$$

So, according to this criterion, the central charge can be any linear combination of n^3 and n. I have noted that any such expression can be reduced to the canonical form of the Virsoro central charge above.

I will now show that the Virasoro algebra has an interesting and restrictive representation theory. We will use the restrictions that come from this theory to compute critical exponents. To set up this analysis, I will define the abstract properties of Virasoro representations. The leading eigenstate of the transfer matrix should be represented by a scale-invariant state $|0\rangle$. This state satisfies

$$
L_0|0\rangle = 0 \qquad L_n|0\rangle = 0 \quad n > 0
$$

More generally, each representations of the Virasoro algebra is built on a *primary conformal state* $|h\rangle$ satisfying

$$L_0 |h\rangle = h|h\rangle \qquad L_n|h\rangle = 0 \text{ for } n>0$$

Such a state is created by a local operator $\mathcal{O}(z)$ with the scaling law

$$\mathcal{O}(z) \rightarrow \lambda^{-h}\mathcal{O}(z/\lambda)$$

Recall that L_{-1} is a translation and so should not annihilate $|h\rangle$. Considering also the transformations on $\overline{z}$, we should write the scaling property of a local operator more precisely as

$$\mathcal{O}(z,\bar{z}) \longrightarrow \lambda^{-h}\bar{\lambda}^{-\bar{h}}\mathcal{O}\left(\frac{z}{\lambda},\frac{\bar{z}}{\bar{\lambda}}\right)$$

The scaling dimension of the operator is

$$D_{\mathcal{O}} = h + \bar{h}$$

Under a rotation,

$$z \rightarrow z e^{i\phi} \qquad \bar{z} \rightarrow \bar{z} e^{-i\phi} \qquad \mathcal{O} \rightarrow e^{i(h-\bar{h})\phi}\mathcal{O}$$

so $(h-\overline{h})$ is the *spin* of the operator.

In the Gaussian model, $c = 1$, there exist operators of any possible positive dimension. Recall that the 2-point correlation function of the Gaussian model is the Green's function of the Laplace equation. In 2 dimensions, this is

$$\langle S(x)\, S(0)\rangle = -\frac{1}{2\pi} \log |x|$$

Then, for any real value of α, we can construct the operator

$$e^{i\alpha S(x)}$$

The dimension of this operator is found from

$$\langle e^{i\alpha S(x)}\, e^{-i\alpha S(0)}\rangle = \exp\left[\alpha^2 \langle S(x)\, S(0)\rangle\right]$$

$$= \frac{1}{|x|^{\alpha^2/2\pi}}$$

Then this operator has the conformal dimensions

$$D_{\mathcal{O}} = \frac{\alpha^2}{4\pi} \qquad h = \bar{h} = \frac{\alpha^2}{8\pi}$$

I will now argue that, if $c < 1$, it is not possible to construct operators with any values of h. In fact, only specific values of c are permitted, and, for these values of c, only specific values of h are possible. The criterion that I will use is *unitarity*, that is, the requirement that all states in the Virasoro representation have positive norm. In statistical mechanics, the transfer matrix is built on a positive-norm Hilbert space, so statistical mechanics models can only contain unitary representations of the transfer matrix.

To study the implications of unitarity, we begin from the state $|h\rangle$, normalized to

$$\langle h | h \rangle = 1$$

Other states in the Virasoro representation can be constructed by applying the operators L_{-1}, L_{-2}, and so forth, to $|h\rangle$. At *level 1*, there is one state, $L_{-1}|h\rangle$. At *level-2*, there are two states,

$$L_{-2}|h\rangle \qquad (L_{-1})^2|h\rangle$$

The norm of these states, and the states at higher levels, can be computed using the fact that

$$L_n|h\rangle = 0 \qquad n > 0$$

together with the Virasoro commutators. It is permissible to find zero norm for some state. Then that state and the higher level states built from it drop out of the representation. However, it is inconsistent with unitarity to find a state with *negative* norm.

For the state at level 1, we find

$$\| L_{-1}|h\rangle \|^2 = \langle h| L_1 L_{-1}|h\rangle = \langle h| [L_1, L_{-1}]|h\rangle$$

$$= \langle h| 2L_0|h\rangle = 2h \, \langle h|h\rangle$$

This implies that we must have $h > 0$ for unitarity. Analyze the norms of states at level 2; we must compute the matrix elements

$$\langle h| L_2 L_{-2}|h\rangle \qquad \langle h| L_1^2 L_{-1}^2|h\rangle \qquad \langle h| L_1^2 L_{-2}|h\rangle$$

Here are the computations:

$$\langle h|L_2L_{-2}|h\rangle = \langle h|\,[L_2,L_{-2}]\,|h\rangle$$

$$= \langle h|\; 4L_0 + \frac{c}{12}\cdot 6 \;|h\rangle = 4h + c/2$$

$$\langle h|\, L_1^2 L_{-1}^2 \,|h\rangle = \langle h|L_1\left(2L_0 L_{-1} + L_{-1}\, 2L_0\right)|h\rangle$$

$$= \left[2(h+1) + 2h\right] \langle h|L_1 L_{-1}|h\rangle = 4h(2h+1)$$

and, finally,

$$\langle h|L_1^2 L_{-2}|h\rangle = \langle h|\, L_1\cdot 3L_{-1}\,|h\rangle = 6h$$

Thus, an arbitrary level 2 state

$$(\alpha L_{-2} + \beta L_{-1}^2)\,|h\rangle$$

has the norm

$$(\alpha\ \beta)^* \begin{bmatrix} 4h + c/2 & 6h \\ 6h & 4h(2h+1) \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

If the determinant of the matrix is negative, the matrix has a negative eigenvalue and this generates a state of negative norm. That determinant is

$$32\left[h^3 - \frac{5-c}{8}h^2 + \frac{c}{16}h\right]$$

We can rewrite this as

$$32h\left(\left(h-\frac{\sqrt{c}}{4}\right)^2+h\frac{(\sqrt{c}-1)(\sqrt{c}+5)}{8}\right)$$

This equation gives no obstruction to unitarity if $c \geq 1$. However, if $c < 1$, there are restrictions.

Consider, for example, the case of $c = \frac{1}{2}$. The determinant becomes

$$32\left(h^3-\frac{9}{16}h^2+\frac{1}{32}h\right)$$

$$= \quad 32h\left(h-\frac{1}{16}\right)\left(h-\frac{1}{2}\right)$$

Then, no value of h between $h = \frac{1}{16}$ and $h = \frac{1}{2}$ can give a unitary representation.

Viktor Kac (of *Kac-Moody algebras*) computed the determinant of inner products of Virasoro states at every level. He showed that, in general, the determinant factorizes into terms

$$\left(h-h_{pq}(c)\right)$$

where $h_{pq}(c)$ is a definite function of c depending on integers p and q. At level 2, it is straightforward to solve the quadratic equation and write

$$32h\left(h-\frac{5-c}{16}+\frac{[(25-c)(1-c)]^{1/2}}{16}\right)\left(h-\frac{5-c}{16}-\frac{[(25-c)(1-c)]^{1/2}}{16}\right)$$

Kac writes this in the form

$$32h\left(h-\left[h_0+\left(\frac{\alpha_+ + 2\alpha_-}{2}\right)^2\right]\right)\left(h-\left[h_0+\left(\frac{2\alpha_+ + \alpha_-}{2}\right)^2\right]\right)$$

where

$$h_0 = -\frac{1-c}{24} \qquad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$$

Kac showed that, at higher levels, the zeros of the determinant are given by

$$h_{pq}(c) = h_0 + \left(\frac{p\alpha_+ + q\alpha_-}{2}\right)^2$$

giving a term $(h - h_{pq}(c))$ in the determinant formula at level $N = pq$.

A given value of h is eliminated by unitarity if the Kac determinant is negative at *any* level. This is an infinite set of constraints that potentially has the power to remove any value of h. In fact, it can be seen that, for $c < 1$, every value of $h > 0$ is eventually included in an interval where the determinant is negative or at the boundary of that interval where the determinant is zero. To avoid a violation of unitarity, we need two conditions.

First, we need that the boundaries of intervals should repeat, and no new ones should be found, as the level N increases. This requires that the parameters α_+ and α_- should be commensurate. Since $0 < c < 1$, this requires,

$$\frac{\alpha_-}{\alpha_+} = \left(\begin{matrix}\text{ratio of}\\ \text{integers}\end{matrix}\right) = -\frac{m}{m+1}$$

Then, for example

$$((m-1)\alpha_+ + \alpha_-)^2 = (\alpha_+ + m\alpha_-)^2$$

and the boundary values $h_{pq}(c)$ do repeat at higher levels. This is a restriction on the value of c,

$$\frac{\sqrt{25-c}-\sqrt{1-c}}{\sqrt{25-c}+\sqrt{1-c}} = \frac{m}{m+1}$$

which implies

$$\sqrt{25-c} = (2m+1)\sqrt{1-c}$$

Thus, the Virasoro representations are unitary only for a discrete series of values of c,

$$c = 1 - \frac{6}{m(m+1)}$$

converging on $c = 1$ as $m \to \infty$. The first nontrivial value is $c = \frac{1}{2}$.

Second, the value of h must be one of the $h_{pq}(c)$. Then the higher-level states that would potentially give negative norm actually have zero norm and do not exist. Inserting the discrete series for c into the expression for $h_{pq}(c)$, the dimensions h in the various discrete series models are predicted to be

$$h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

with $q \leq p < m$ before the conditions begin to repeat.

These deep considerations give a series of critical points in 2 dimensions, with, in each case, the scaling dimensions of the operators in the scale-invariant model.

The first of these models is the theory with $c = \frac{1}{2}$. The operator dimensions are

$$h_{1,1} = \frac{(4-3)^2-1}{48} = 0 = h_{2,3}$$
$$h_{2,1} = \frac{(8-3)^2-1}{48} = \frac{1}{2} = h_{1,3}$$
$$h_{1,2} = \frac{(4-6)^2-1}{48} = \frac{1}{16}$$

This model can be identified with the 2-dimensional Ising model. Onsager's solution of the model demonstrated the equivalence of the model to a model of a free fermion. The scale invariant theory of a free fermion in 2 dimensions has $c = \frac{1}{2}$. The fermion is an operator of spin $\frac{1}{2}$,

$$\psi \quad (h,\bar{h}) = (\tfrac{1}{2}, 0) \quad \oplus \quad \bar{\psi} \quad (h,\bar{h}) = (0, \tfrac{1}{2})$$
$$D_\psi = \frac{1}{2}$$

The spin S_i corresponds to an operator with

$$(h,\bar{h}) = (\tfrac{1}{16}, \tfrac{1}{16}) \qquad D_S = \frac{1}{8}$$

These values give the results previously cited on the critical exponents of the 2-dimensional Ising model. At the critical point, the fermion-fermion correlation function scales as

$$\langle \psi(x)\, \psi(0) \rangle \sim \frac{1}{z^1}$$

while the spin-spin correlation function scales as

$$\langle S(x)\, S(0) \rangle \sim \frac{1}{|x|^{1/4}}$$

The value of η is obtained from

$$D_S = A = \frac{d-2+\eta}{2} \Rightarrow$$

giving

$$\eta = \frac{1}{4}$$

The operator $\mathcal{O}_t$ is the mass operator for the fermion,

$$\bar{\Psi}\psi \qquad (h,\bar{h}) = (\tfrac{1}{2},\tfrac{1}{2}) \qquad D_{\bar{\Psi}\psi} = 1$$

This gives

$$b_t = d - D_{\bar{\Psi}\psi} = 2-1 = 1$$

or

$$\nu = 1$$

Finally,

$$\beta = \frac{(d-2+\eta)\,\nu}{2} = \frac{1}{8}$$

as was found in Yang's calculation.

Cardy, *Huse*, and *Friedan and Shenker* identified the higher models in this series. The case $m = 4$, $c = \frac{7}{10}$ describes a tricritical point in the Ising model.

The case $m = 5$, $c = \frac{4}{5}$, is the *3-state Potts model*. The N-state Potts model is a model in which the spin S_i can take any of N values, with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta(S_i, S_j)$$

The case $N = 2$ is the Ising model. Mean field theory predicts that, for $N > 2$, the Potts model has a discontinous transition to the ordered phase. However, *Fisher and Straley* showed from high temperature series expansions that the case $N = 3$ actually has a critical point. The case $N = 4$ also has a critical point, with discontinuous transitions for higher N. The $N = 4$ case gives a system with $c = 1$. The case $m = 6$, $\frac{c=6}{7}$ gives a tricritical point in the 3-state Potts model.

The cases $m = 7, 8, \cdots$ are cases of the *Restricted Solid-on-Solid* model solved by Andrews, Baxter, and Forrester. The limit $m \to infty$ of this model is equivalent to a theory of a free boson. Each model with $m > 3$ contains a relevant operator that induces a flow to a model in the series with smaller c.

This brings us to the end of this course in Statistical Mechanics. Statistical Mechanics is a very rich subject, and there is much more to learn about it. Some subjects that I have not adequately discussed are

- the implications of crystal structure and symmetries (*solid-state physics*)
- the quantum theory of many-particle interactions, and effects such as superconductivity that arise in this setting
- more powerful methods, such as the Yang-Baxter equation, for the exact solution of statistical models
- time-dependent phenomena near equilibrium in order media, for example, the hydrodynamics of superfluids
- time-dependent phenomena far from equilibrium
- systems with tensor and other exotic order parameters, such as superfluid He^3
- systems with disorder, including spin glasses and amorphous materials

I hope that this course has given you a secure foundation from which to continue your exploration of this fascinating subject.