May 11

Superfluids

Superfluids are systems in which interacting bosons condense into a macroscopically occupied quantum state. I have explained already that superfluids have the same symmetry and the same transformation properties of the order parameter as XY ferromagnets. As such, they provide an interesting application of Landau theory with continuous symmetry. We will see that the some fascinating physical properties that follow from the general formal considerations explained in the previous lecture.

To begin, I will discuss the order parameter of a superfluid in more detail. Systems with identical particles are described in quantum mechanics by *second quantization.* We define *creation* and *annihilation* operators $a_i^\dagger$, a_i such that

- $a_i^\dagger$ creates a particle in the single-particle quantum state i
- a_i destroys a particle in the state i
- there is a state with no particles, $|0\rangle$, that satisfies $a_i|0\rangle = 0$

For identical bosons, the operators a_i, $a_i^\dagger$ satisfy the algebra of harmonic oscillator raising and lowering operators

$$[a_i, a_j^\dagger] = \delta_{ij}$$

For identical fermions, the creation and annihilation operator satisfy the same relation with an *anticommutator* on the left-hand side. The commutation relation can be rewritten as

$$a_i a_i^\dagger = a_i^\dagger a_i + 1$$

We can now set up normalized states with 1, 2, $\cdots$ particles in the state i. The 1-particle state is

$$|1\rangle = a_i^\dagger |0\rangle$$

We can check the normalization from the equation above,

$$\| |1\rangle \|^2 = \langle 0| a_i a_i^\dagger |0\rangle = \langle 0| a_i^\dagger a_i + 1 |0\rangle = \langle 0|0\rangle = 1$$

The 2-particle state is given by

$$|2\rangle = \frac{1}{\sqrt{2}} (a_i^\dagger)^2 |0\rangle$$

The normalization of this state can be check using two applications of the commutation relation,

$$\left\| \frac{(a_i^\dagger)^2|0\rangle}{\sqrt{2}} \right\|^2 = \frac{1}{2} \langle 0| a_i^2 a_i^{\dagger 2} |0\rangle$$

$$= \frac{1}{2} \langle 0| a_i \left[(a_i^\dagger)^2 a_i + 2a_i^\dagger \right] |0\rangle$$

$$= \frac{1}{2} \cdot 2 \langle 0| a_i a_i^\dagger |0\rangle = 1$$

The normalized n-particle state is

$$|n\rangle = \frac{(a_i^\dagger)^n}{\sqrt{n!}} |0\rangle$$

It follows from this formalism that

$$a_i^\dagger a_i \; |n\rangle = n \; |n\rangle$$

so that $N_i = a_i^\dagger a_i$ is the number of particles in the state i. Also,

$$\langle n+1 | a_i^\dagger | n \rangle = \langle n | a_i | n+1 \rangle = \sqrt{n+1}$$

In addition to these states with definite particle number, we can define states with an indefinite number of particles and a well-defined quantum mechanical phase. This is called a *coherent state*. Let α be a complex number, and define

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha a_i^\dagger} |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (a_i^\dagger)^n |0\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \end{aligned}$$

The state $|\alpha\rangle$ has the following properties: First, it is normalized

$$\begin{aligned} \langle \alpha | \alpha \rangle &= e^{-|\alpha|^2} \sum_{m,n} \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \langle m | n \rangle \\ &= e^{-|\alpha|^2} \sum_{m} \frac{(\alpha^*\alpha)^m}{m!} = e^{-|\alpha|^2} e^{|\alpha|^2} = 1 \end{aligned}$$

Second, it is an eigenstate of the annihilation operator a_i,

$$\begin{aligned} a_i |\alpha\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} a_i |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n} \, |n-1\rangle \end{aligned}$$

or, finally,

$$a_i|\alpha\rangle = \alpha\,|\alpha\rangle$$

The expected number of particles in the coherent state is

$$\langle\alpha|\,a_i a_i^\dagger\,|\alpha\rangle = |\alpha|^2\,\langle\alpha|\alpha\rangle = |\alpha|^2$$

This state has quantum fluctuations in the number of particles. The probability of finding n particles in a coherent state is

$$\mathrm{Prob}(n) = e^{-|\alpha|^2}\,\frac{|\alpha^2|^n}{n!}$$

which is a *Poisson distribution* with $\langle n\rangle = |\alpha|^2$. In quantum mechanics, the phase of a wavefunction is conjugate to the particle number, so there is a Heisenberg uncertainty priniciple

$$\Delta\phi\,\Delta n \geq \frac{\hbar}{2}$$

However, for a *macroscopically occupied* state with $\langle n\rangle \sim N$, the fluctuations in n are of the order of $\sqrt{N}$. Then it is essentially equivalent to work with quantum states of definite particle number or of definite phase. The states of definite phase are a more convenient description of the superfluid.

To describe the space- and time-dependence of macroscopic quantum condensates, a little more formalism is useful. Let $\mathcal{H}$ be the single-particle Hamiltonian, and let the eigenstates of $\mathcal{H}$ be $\psi_i(x)$,

$$\mathcal{H}\,\psi_i(x) = E_i\,\psi_i(x)$$

Now define the operator on the second-quantized Hilbert space

$$\Phi(x,t) = \sum_i \psi_i(x)\, e^{-iE_i t}\, a_i$$

This operator is called a *quantum field.* It is a local operator associated with the point (t, x). It obeys the Schrödinger equation

$$i\frac{\partial}{\partial t}\Phi = \mathcal{H}\Phi$$

as its Heisenberg equation of motion. The quantum field annihilates one particle, a particle located at x at time 0. The form of $\Phi(x)$ at $t = 0$ is actually independent of the basis of single-particle wavefunctions that is used; however, the time-dependence is more complicated in a general basis.

We can now construct spatially-dependent coherent states as the eigenstates of $\Phi(x)$,

$$\Phi(x)\,|\Psi\rangle = \Psi(x)\,|\Psi\rangle$$

For example, if we construct the state $|\alpha, i\rangle$ such that

$$\Phi(x)\,|\alpha,i\rangle = \alpha\,\psi_i(x)\,|\alpha,i\rangle$$

then this is a coherent state with expected number of particles $\langle n\rangle = |\alpha^2|$ in the single-particle wavefunction $\psi_i(x)$.

A thermodynamically large system with macroscopic occupation of the zero-momentum state, as we found in Bose-Einstein condensation, is described by this formalism in the following way: The zero-momentum state has the single-particle wavefunction

$$\psi_0 = \frac{1}{\sqrt{V}}$$

For the coherent state $|N\rangle$ in which this is occupied by N particles, then,

$$\langle N | \Phi(x) | N \rangle = \sqrt{\frac{N}{V}}$$

Note that this is an intensive quantity, a density like the magnetization density $M(x)$. We can use this observation to define the order parameter of a superfluid,

$$\Phi(x) = \frac{\mathrm{tr}\, \Phi(x)\, e^{-\beta H}}{Z}$$

As for other systems with spontaneously broken symmetry, this expectation value is trivially zero when averaged with the full canonical ensemble but is nonzero when averaged in the states built on one macroscopically ordered ground state.

Notice that this definition does not assume free particles and can be applied to a system of interacting bosons. The order parameter $\Phi(x)$ is a complex number, and the formalism is invariant under global phase rotations, so this is a theory with the symmetry of the XY ferromagnet.

It is odd that the order parameter of a superfluid is the expectation value of an operator that changes particle number. Yang gave this situation a special name: *off-diagonal long-range order.* However, this description is very effective in describing the properties of superfluids. It seems weird only because quantum mechanics itself is weird.

Given this symmetry and this order parameter, we can construct the Landau free energy for a superfluid

$$G = \int d^3x \left\{ \frac{\ell}{2} |\vec{\nabla} \Phi|^2 + \frac{1}{2} a (T - T_c) |\Phi|^2 + \frac{1}{4} b |\Phi|^4 \right\}$$

From our general analysis, we know that this Landau free energy predicts that, just below $T = T_C$,

$$\Phi \sim (T_c - T)^{1/2}$$

For the free Bose-Einstein gas, we computed the macroscopic occupation of the single-particle ground state in Bose-Einstein condensation to behave as

$$N_s \sim (T_c - T)^1$$

Since $\Phi = (N_s/V)^{1/2}$, we see that these results are nicely consistent. However, I will also use this Landau theory to to describe the behavior of *interacting* boson gases such as He^4 and very low temperature condensates of heavy atoms. He^4 liquifies at about 4°K. A superfluid condensate first appears at about 2°K, accompanied by a sharp feature in the specific heat called the λ*-point.*

Systems with a nonzero order parameter Φ have the amazing property that fluid flows frictionlessly. This is the defining property of a *superfluid.* I will now give two arguments for this frictionless fluid flow.

In single-particle quantum theory, the particle number current is

$$\vec{j} = \frac{-i\hbar}{2m}\left[\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi\right]$$

For a particle in the wavefunction

$$\psi = \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{V}}$$

the current is

$$\vec{j} = \frac{1}{V} \cdot \frac{\hbar \vec{p}}{m} = n \cdot \vec{v}$$

where $n = N/V$, with $N = 1$ in this case, and $\vec{v}$ is the particle velocity. In a many-particle system, the *current operator* is constructed from the *quantum field* by

$$\vec{J} = \frac{-i\hbar}{2m} \left[\Phi^{+} \vec{\nabla} \Phi - (\vec{\nabla} \Phi)^{+} \Phi \right]$$

If $\Phi(x) = \Phi_0$, a constant, $\vec{J} = 0$. However, if

$$\Phi(x) = \Phi_0 \, e^{i \vec{p} \cdot \vec{x}}$$

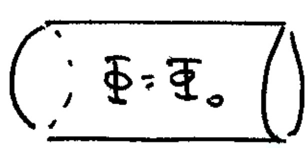
then

$$\vec{J} = |\Phi_0|^2 \, \frac{\hbar \vec{p}}{m} = n \cdot \vec{v}$$

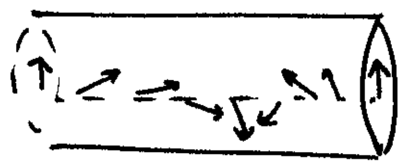
with

$$n = \frac{N}{V}$$

With this in mind, imagine that we have a pipe filled with superfluid

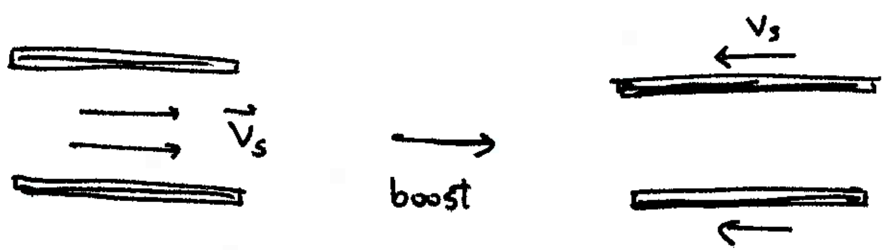


Smoothly twist the phase of $\Phi(x)$ from one end of the pipe to the other. If the pipe is periodically connected, we can imagine giving the phase of $\Phi(x)$ one or more full twists



This configuration of $\Phi(x)$ has nontrivial topology. Indeed, the configuration in which the phase changes as gradually as possible is *topologically stable.* This configuration cannot relax to a state in which $\Phi(x)$ is constant by smooth deformations. The state with a gradient of the phase of $\Phi(x)$ is nonzero is a state with $\vec{J} \neq 0$. This is the state of *lowest free energy* in its topological class. This state is then absolutely stable. The fluid flow, and it dissipates no energy in doing so.

Here is a second, very different argument for superfluidity, due to Landau. Consider a pipe in which superfluid is flowing at velocity v_s By a Galilean boost, examine this situation in a frame in which the superfluid is at rest and the pipe is moving backward at velocity v_s.



The process in which the fluid is slowed down by friction is described microscopically by the production of excitations along the wall that move more slowly than the fluid. The energy of the coherent motion of the fluid turns into the energy of these excitations. In the boosted frame, the wall creates these excitations in the stationary fluid, and they move backwards along with the wall.

At low temperatures, the excitations of the fluid are the elementary quantum excitations that are possible in the medium. In an ordinary fluid, these low-energy excitations involve single fluid particles. The excitation energies are

$$E(p) = \frac{p^2}{2m}$$

An excitation with energy and momentum $(E, \vec{p})$ in the boosted frame has, in the lab frame, energy

$$E_{lab} = E(p) - \vec{v}_s \cdot \vec{p}$$

For a single-particle excitation, this is *negative* for sufficiently small $\vec{p}$. Then the overall fluid flow loses energy to these excitations.

However, we argued that, in an ordered medium, the lowest-energy excitations are the Goldstone bosons, called *phonons* in this context. Their energy spectrum is

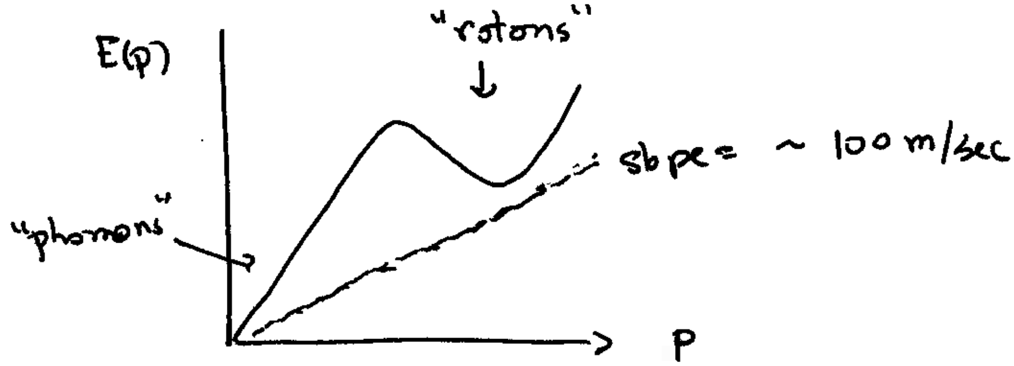
$$E(p) = c_s |\vec{p}|$$

This translates, in the lab frame, into an energy of

$$E_{lab} = c_s |\vec{p}| - \vec{v}_s \cdot \vec{p}$$

which is positive as long as $v_s < c_s$.

By neutron scattering from superfluid He^4, it is possible to map the spectrum of elementary excitations. The spectrum that is found is the following:



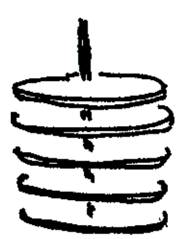
The lowest-energy excitations are the phonons. At higher p, one finds also excitations called *rotons*. If v_s is less than the slope of the dotted line, all single-particle excitations increase the energy of the flowing superfluid in the lab frame, and so the fluid flow does should not decay. In practice, the superflow more typically breaks down at a lower velocity of a few m/sec.

The phonons are localized bursts of superfluid velocity. The rotons must be orthogonal to these and may be thought of as quantized vortex rings. Their spectrum is

$$E(p) = \Delta + \frac{(p-p_0)^2}{2m_*} \qquad m_* \approx 0.16\, m_{He^4}$$

The thermodynamics of the superfluid phase of He^4 can be described by considering this system to be a two-component boson gas of phonons and rotons. Momentum is carried both by the superfluid and by the gas of rotons. The former component is frictionless; the latter has normal viscosity, depending on the density of excitations. This leads to a *two-fluid* description of fluid flow in He^4, with separate superfluid and normal fluid velocities.

The two-fluid description of low-temperature He^4 was demonstrated in a 1946 classic experiment. Andronikashvili constructed a torsion pendulum consisting of parallel copper disks.



The frequency of the pendulum is higher in vacuum that if the pendulum is immersed in a fluid, because the disks drag fluid along with them, increasing their inertia. Andronikashvili showed that the frequency of the pendulum in liquid He^4 increases as the temperature decreases, returning to the vacuum value as $T \to 0$. In this limit, there is no more normal fluid, and the superfluid does not rotate with the disks.

The superfluid component of the flow can be described by a form of $\Phi(x)$ in which the magnitude remains close to the minimum of $G[\Phi]$ while the phase varies. Then

$$\Phi(x) \approx \Phi_0 e^{i\varphi(x)}$$

This gives

$$\vec{J} = n \cdot \frac{\hbar \vec{\nabla} \varphi}{m}$$

or

$$\vec{V}_s = \frac{\hbar}{m} \vec{\nabla} \varphi$$

This expression implies

$$\vec{\nabla} \times \vec{V}_s = 0$$

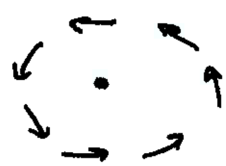
that is, the flow is *irrotational*. This raises an interesting question: What if we set up a bucket of liquid He^4, start the fluid spinning, and the cool it below the λ-point?

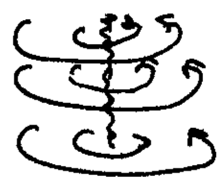
The answer is given by the *vortex* solutions in the Landau theory of the XY model discussed in the previous lecture. Theses objects have

$$\Psi = \Psi_0 f(r) e^{i\phi}$$

where $f(r) \to 1$ as r goes outside of the vortex core. In superfluid He^4, the radius of a vortex is a few atomic spacings. Outside of the core,

$$\vec{v}_s = \frac{\hbar}{m} \vec{\nabla} \phi = \frac{\hbar}{m} \frac{\hat{\phi}}{r}$$





Note that the *vorticity* of this object is *quantized*:

$$\oint d\vec{x} \cdot \vec{v}_s = \frac{2\pi\hbar}{m}$$

the rotating superfluid contains an array of quantized vortices sufficient to account for the angular momentum that the rotating fluid had above the λ-point.

When a superflow goes through an aperture, it can set up a large vortex with macroscopic vorticity. These objects can nucleate a return to the normal state if the fluid velocity becomes too high.

There is much more to say about superfluidity, but I will now go on to another important topic, *superconductivity*. This is the frictionless flow of electric current in metals at very low temperature. The mechanism of superconductivity is the formation of $e^- e^-$ bound states build from two electrons of opposite spin near the Fermi energy. These bound states are bosons. At low temperature, they form a macroscopic Bose-Einstein condensate. *Cooper* showed that the effect of any small attractive interaction between electrons is amplified near the Fermi surface. From this observation, *Bardeen, Cooper, and Schrieffer* built a complete theory of the attraction, binding, and condensation of electron pairs. I strongly recommend to you the close study of this theory. In this courses, I will only have time to discuss those aspect of superconductivity that are accessible using a phenomenological description based on Landau theory.

Any metal can become superconducting at sufficiently low temperature. The source of the attractive interaction is comes from lattice vibrations (phonons), as one electron is deflected by the lattice vibration caused by another. Typically, the transition temperature is *very* low. Some metals that become superconducting at temperatures that are not prohibitively low are

$$\text{Al} \quad 1.2^\circ K \qquad \text{Zn} \quad 0.9^\circ K \qquad \text{Pb} \quad 7^\circ K \qquad \text{Nb} \quad 9.3^\circ K$$

To build a phenomenological description of superconductivity, let $\Phi(x)$ be the quantum field of the $e^- e^-$ bound states, called *Cooper pairs.* Assign each pair an effective mass m_* and a charge $e_* = 2e$. The fact that the condensing particles are charged is essential. For a complete description of the superconducting state, we must couple $\Phi(x)$ to the electromagnetic field. To describe this coupling, we should modify the Landau free energy so that the variational equation contains the kinetic term found in Schrödinger equation in the presence of a nonzero vector potential,

$$-\frac{\hbar^2}{2m_*}\left(\vec{\nabla} + i\frac{e_*}{\hbar c}\vec{A}\right)^2$$

An alternative way of describing this requirement is that we make the Landau free energy *gauge invariant* with respect to the gauge symmetry of electromagnetism. This gives the *Landau-Ginzburg free energy*

$$G = \int d^3x \left\{ \frac{\hbar^2}{2m_*}\left|\left(\vec{\nabla} + i\frac{e_*}{\hbar c}\vec{A}\right)\Phi\right|^2 + \frac{1}{2}a(T-T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right\}$$

This coupling is invariant to the gauge transformation

$$\Phi(x) \to e^{i\frac{\alpha(x)}{\hbar}}\Phi(x) \qquad \vec{A} \to \vec{A} - \frac{c}{e_*}\vec{\nabla}\alpha(x)$$

as required.

The current associated with Φ is

$$\vec{J}_{EM} = i \frac{e_* \hbar}{2m_*} \Phi^* \vec{D} \Phi - (\vec{D}\Phi)^* \Phi$$

where

$$\vec{D}\Phi = (\vec{\nabla} + i \frac{e_*}{\hbar c} \vec{A}) \Phi$$

The current $\vec{J}_{EM}$ is gauge-invariant.

The gauge symmetry allows us to remove the phase of $\Phi(x)$ by a gauge transformation. If, also, the magnitude of $\Phi(x)$ stays at the minimum Φ_0 of $G[\Phi]$, we find

$$\vec{J}_{EM} = - \frac{e_*^2}{m_* c} |\Phi_0|^2 \vec{A}$$

This relation is called the *London equation.*

The Landau-Ginzburg free energy predicts that the superfluid order parameter turns on as

$$|\Phi| \sim (T_c - T)^{\frac{1}{2}}$$

for T just below T_C, and that below the transition there is superfluidity and frictionless flow of electric current. However, there are additional remarkable properties that follow from the coupling to $\vec{A}$.

Starting from the Maxwell equation

$$-\frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}_{EM}$$

consider a static configuration, so that

$$\frac{\partial \vec{E}}{\partial t} = 0$$

and take the curl. Then

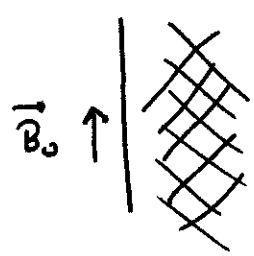
$$\nabla \times (\vec{\nabla} \times \vec{B}) = \frac{4\pi}{c} \vec{\nabla} \times \vec{J}_{EM}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -\frac{4\pi}{c} \frac{e_*^2}{m_* c} \vec{\nabla} \times \vec{A} \, |\Phi_0|^2$$

The second line uses the London equation. Since $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{A} = \vec{B}$, this becomes

$$\nabla^2 \vec{B} = \frac{4\pi e_*^2}{m_* c^2} |\Phi_0|^2 \vec{B}$$

An implication of this equation is that, if $\vec{B} = \vec{B}_0$ on the surface of a superconductor,



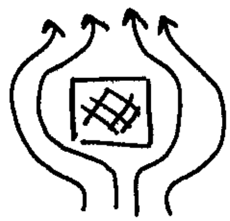
then $\vec{B}$ falls off into the interior as

$$\vec{B} = \vec{B}_0 \, e^{-z/\lambda}$$

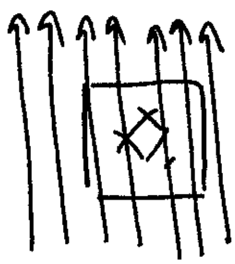
The parameter λ is called the *penetration depth.* It is given by

$$\lambda = \left[\frac{4\pi e_*^2 |\Phi_0|^2}{m_* c^2} \right]^{-1/2}$$

Thus, superconductors expell magnetic field. This is called the *Meissner effect*



The converse of this statement is that magnetic fields destabilize superconductivity. There is a *critical field* above which a metal cannot become superconducting. I will now compute this field, which I will refer to in terms of the externally applied field $\vec{H}$. To do this, write the Gibbs free energy—including the magnetic field energy—for the normal and superconducting phases. In the normal phase, $G = 0$ from the superconducting order parameter, but there is a magnetic field energy



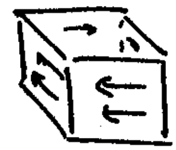
$$G = \frac{H^2}{8\pi} \cdot \text{Volume}$$

In the superconducting phase, there is the usual contribution from spontaneous symmetry breaking.

$$G = (\text{Volume}) \cdot \left[-\frac{a}{2}(T_c - T)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right]$$

$$= (\text{Vol.})\left\{ -\frac{1}{4}\frac{a^2}{b}(T_c - T)^2 \right\} \quad \text{for} \quad \Phi = \left[\frac{a(T_c - T)}{b} \right]^{1/2}$$

But also, the superconductor must have a magnetic moment from the currents that flow to expel the magnetic field.



If the magnetic field in the superconductor is zero, then

$$\vec{M} = -\frac{\vec{H}}{4\pi}$$

and so this contribution to the energy is

$$G = \text{Vol.} \cdot (-\vec{H} \cdot \vec{M}) = (\text{Vol.}) \cdot \frac{H^2}{4\pi}$$

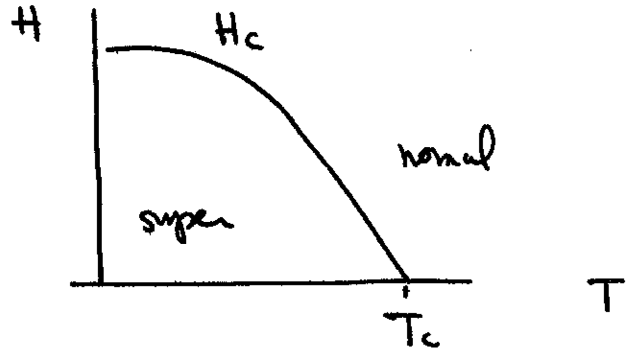
Balancing these effects, we find the inequality required for superconductivity to be stable.

$$\frac{H^2}{8\pi} \leq \frac{1}{4} \frac{a^2}{b} (T_c - T)^2$$

Equality here gives the *critical field*

$$H_c = \left[\frac{2\pi a^2}{b}\right]^{\frac{1}{2}} (T_c - T)$$

Above $H = H_C$, superconductivity is destabilized. At $H = H_C$, there is a *discontinuous* phase transition to the normal metal.



There is still more to say about the interaction of superconductivity and magnetic fields. To introduce this, I will first point out that we now have two distinct length scales that arise from the phenomenological theory. The first is $\xi(T)$, the correlation length of $\Phi(x)$, that is, the distance over which a deviation of $|\Phi|$ from its minimum relaxes to Φ_0. The second is $\lambda(T)$, the distance over which a $\vec{B}$ field in the superconductor relaxes to 0. By varying the properties of the material, it is possible to find metals with very different ratios of ξ to λ. We may define:

Type I superconductors $\qquad \xi(T) > \lambda(T)$

Type II superconductors $\qquad \xi(T) < \lambda(T)$

The phase diagram shown above is typical for Type I superconductors, but in Type II materials another thermodynamic phase appears.

To explain this, I will generalize the vortex solution that we found in superfluids to the case of superconductivity. We now need to minimize

$$G = \int d^3x \left\{ \frac{\hbar^2}{2m_*} \left| \left(\vec{\nabla} + i \frac{e}{\hbar c} \vec{A} \right) \Phi \right|^2 + \frac{a}{2}(T-T_c)|\Phi|^2 + \frac{b}{4}|\Phi|^4 \right\}$$

for field configurations in the topological class of the vortex. In the XY model and the superfluid, we saw earlier that the vortex has a free energy that is logarithmically divergent at large distances, since if

$$\Phi = \Phi_0 f(r) e^{i\phi}$$

then

$$\vec{\nabla}\Phi \sim i\Phi_0 \frac{\hat{\phi}}{r}$$

and so

$$G \sim \int^{L} dr\, r\, \frac{1}{r^2} \sim \log L$$

In the case of a superconductor, we use the freedom to turn on a nonzero $\vec{A}$ to remove this divergent contribution to the energy. If

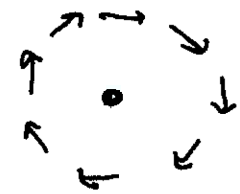
$$\vec{A} \cong -\frac{\hbar c}{e_*}\frac{1}{r}\hat{\phi} \qquad \text{for large } r$$

then

$$D\Phi = \left(\vec{\nabla} + \frac{ie_*}{\hbar c}\vec{A}\right)\Phi \sim i\Phi_0\frac{\hat{\phi}}{r} - i\frac{\hat{\phi}}{r}\Phi_0$$

and the dangerous kinetic term cancels. It is possible to find a solution in which the density of Gibbs free energy falls to zero exponentially as $r \to \infty$. This solution includes a flow of $\vec{A}$ around the vortex

$$\oint d\vec{x}\cdot\vec{A} = -\frac{2\pi\hbar c}{e_*}$$



Since

$$\oint_{\partial S} d\vec{x}\cdot\vec{A} = \int_S d^2s\ \hat{n}\cdot\vec{B}$$

the vortex then carries a *quantized magnetic flux*

$$\int d^2s\ \hat{n}\cdot\vec{B} = \frac{2\pi\hbar c}{e_*}\cdot m \qquad m = \underline{\text{integer}}$$

It is amazing that this formula involves only fundamental constants, with no phenomenological parameters! In 1961, *William Fairbank*, here at Stanford, verified this prediction and confirmed that $e_* = 2e$, in agreement with the idea that superconductivity is associated with a condensate of electron pairs.

In Type I superconductors, vortices attract one another. Then if a large magnetic field is put through a superconductor, vortices form and then coalesce into a large vortex tube with normal metal in the interior. This gives a mechanism for a sufficiently large $\vec{H}$ to force a transition to the normal state. However, in Type II superconductors, vortices *repel* one another. Then, putting a large magnetic field through a superconductor creates an array of singly-quantized vortices.

It is possible to find the explicit vortex solution in the Type II limit. Here ξ is small, so $f(r)$ goes quickly to 1, but λ is large, so the B field relaxes more slowly. For a cylindrically symmetric situation, the London equation becomes

$$\nabla^2 B_z = \frac{1}{\lambda^2} B_z$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} B_z(r) = \frac{1}{\lambda^2} B_z(r)$$

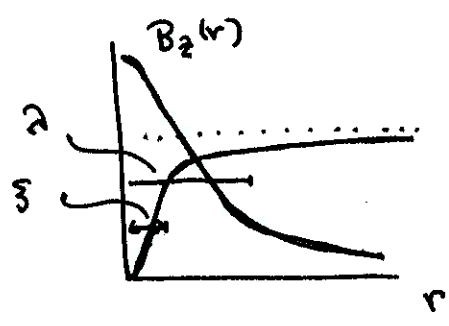
The solution is

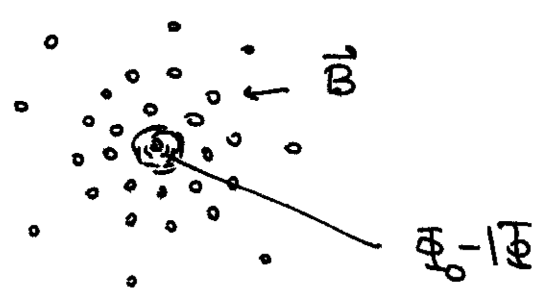
$$B_z = \frac{2\pi\hbar c/e_*}{2\pi\lambda^2} K_0(r/\lambda)$$

where $K_0(z)$ is the modified Bessel function. As $z \to \infty$,

$$K_0(z) \sim \frac{1}{\sqrt{z}} e^{-z}$$

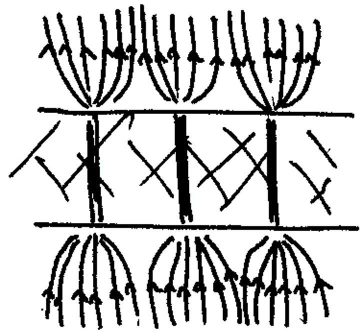
The Bessel function $K_0(z)$ as a log divergence as $z \to \infty$. This is cut off when we enter the small region where $r < \xi$, $f(r) \to 0$. A picture of the vortex is then



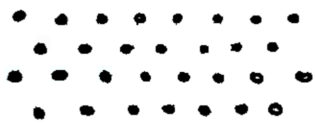


This object is called the *Abrikosov flux tube.*

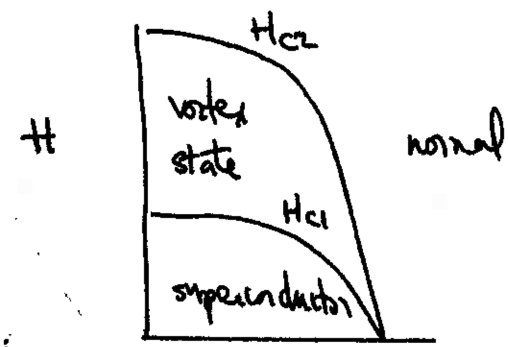
Abrikosov realized that these vortices produce a new thermodynamic phase in the phase diagram of H versus T. At a certain magnetic field H_{C1}, magnetic flux can penetrate a Type II superconductor and form an array of vortices.



This phase is called the *Abrikosov vortex state.* In this phase, magnetic field and superconductivity coexist. The most stable configuration of vortices turns out to be a hexagonal lattice.



At a higher field H_{C2}, the superconductivity is completely destroyed. This leads to a phase diagram of the form



Superconducting magnets operate in the vortex state phase.

Superconductors have many more fascinating properties that we will not have time to discuss in this course. An excellent reference for further reading is Michael Tinkham's book, *Introduction to Superconductivity.*