# Physics 212 - Final Exam

## Solutions

1.) For a relativistic ideal gas, we have

$$\frac{E}{V} = g\,\frac{\pi^2}{30} T^4 \frac{1}{(\hbar c)^3} \cdot \begin{cases} 1 & \text{Bose Einstein} \\ \frac{7}{8} & \text{Fermi Dirac} \\ & (\mu = 0) \end{cases}$$

$$P = \frac{1}{3}\frac{E}{V} \quad \text{in both cases}$$

a) I did not work out $N/V$ in class, but this is readily done

For bosons

$$\frac{N}{V} = g \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta c k} - 1}$$

$$= g \left(\frac{T}{\hbar c}\right)^3 \frac{1}{2\pi^2} \int_0^\infty dk \, \frac{k^2}{e^k - 1}$$

$$= g \frac{1}{2\pi^2} \left(\frac{T}{\hbar c}\right)^3 \Gamma(3)\, \zeta(3)$$

$$= g \frac{1}{\pi^2} \zeta(3) \left(\frac{T}{\hbar c}\right)^3 \qquad \zeta(3) = 1.2021$$

For fermions, this integral is replaced with

$$\int_0^\infty dk \frac{k^2}{e^k+1} = \Gamma(3)\zeta(3)\left(1-\frac{1}{2^2}\right) = 2\cdot\zeta(3)\cdot 3/4$$

Photons, electrons, and positrons all have $g=2$, so

$$\frac{N}{V} = \underbrace{\frac{2}{\pi^2}(1.2021)}_{0.244} \cdot \left(\frac{T}{\hbar c}\right)^3 \cdot \left\{ \begin{array}{ll} 1 & \gamma \\ 3/4 & e^- \\ 3/4 & e^+ \end{array} \right.$$

For $T = 200\,\text{MeV}$

$$\frac{T}{\hbar c} = \frac{200\,\text{MeV}}{197.3\,\text{MeV-fm}} = 1.015 \times 10^{15}/\text{m}$$

$$\left(\frac{T}{\hbar c}\right)^3 = 1.05 \times 10^{45}/\text{m}^3$$

The volume of a 1-m balloon is $\frac{4}{3}\pi R^3 = 4.2\ \text{m}^3$

so the number of photons, electrons, and positrons is

$$N = 1.07 \times 10^{45} \times \left\{ \begin{array}{ll} 1 & \gamma \\ 3/4 & e^- \\ 3/4 & e^+ \end{array} \right.$$

The pressure of a photon gas is

$$P = \underbrace{\frac{\pi^2}{45}}_{0.219} \cdot T \cdot \left(\frac{T}{\hbar c}\right)^3$$

The extra factor of T is

$$200 \text{ MeV} \cdot 1.602 \times 10^{-19} \text{ J/eV} = 3.2 \times 10^{-11} \text{ J}$$

$$= 3.2 \times 10^{-11} \text{ kg m}^2/\text{sec}^2$$

so

$$p = 0.74 \times 10^{34} \text{ newton/m}^2$$

Counting photons, electrons, and positrons, we have:

$$\text{above} \cdot \quad 1 + \frac{7}{8} + \frac{7}{8} = \frac{11}{4}$$

or

$$p = 2.0 \times 10^{34} \text{ newton/m}^2$$

b.) The work done by the balloon in expanding is

$$W = \int p \, dV = pV$$

since p is constant in this process. The result is:

$$W = \frac{11}{4} \frac{\pi^2}{45} \frac{T^4}{(\hbar c)^3} \cdot \frac{4\pi}{3} R^3$$

At the end of the process, the balloon has energy

$$E = \frac{11}{4} \cdot \frac{\pi^2}{15} \frac{T^4}{(\hbar c)^3} \cdot \frac{4\pi}{3} R^3$$

so the total energy absorbed by the balloon must be

$$E_{abs} = \frac{11}{4} \cdot \frac{4\pi^2}{45} \frac{T^4}{(\hbar c)^3} \cdot \frac{4\pi}{3} R^3$$

or $$E_{abs} = \frac{11\pi^2}{45} \frac{T^4}{(\hbar c)^3} \cdot \frac{4\pi}{3} R^3$$

c.) An adiabatic expansion has constant entropy. The entropy of a relativistic gas is

$$S = \frac{4}{3}\frac{E}{T} = V \cdot \frac{11}{4} \cdot \frac{4\pi^2}{45} \left(\frac{T}{\hbar c}\right)^3$$

In this process, the volume increases by a factor $10^3$

Since $\frac{S}{V} \sim T^3$, the temperature must decrease by a factor of 10. Then the final temperature will be

$$T_f = 20 \text{ MeV}$$

or $$T = \frac{200 \text{MeV}}{R}$$ where R is the current radius of the sphere (in m). The work done is computed as follows:

Let $T_i = 200$ MeV $\quad R_i = 1$ m

$T_f = 20$ MeV $\quad R_f = 10$ m

$$W = \int p\,dV = \int_{R=R_i}^{R=R_f} \frac{11}{4}\frac{\pi^2}{45}\frac{1}{(\hbar c)^3} T^4(R)\; 4\pi R^2 dR$$

$$= \frac{11}{4}\frac{\pi^2}{45}\frac{1}{(\hbar c)^3} T_i^4 \int_{R_i}^{R_f} \left(\frac{R_i}{R}\right)^4 4\pi R^2 dR$$

$$= \frac{11}{4}\frac{\pi^2}{45}\frac{1}{(\hbar c)^3} T_i^4\, 4\pi R_i^4 \left(\frac{1}{R_i} - \frac{1}{R_f}\right)$$

$$\underline{W_{adiabatic}} = \frac{11}{4}\frac{\pi^2}{15}\frac{T_i^4}{(\hbar c)^3}\; \frac{4\pi R_i^3}{3}\left(1 - \frac{R_i}{R_f}\right)$$

d.) If the balloon expands at a speed $\frac{dR}{dt}$, the number density of photons changes at the rate

$$n = 0.24\left(\frac{T}{\hbar c}\right)^3 = 0.24\left(\frac{T_i}{\hbar c}\right)^3 \cdot \left(\frac{R_i}{R}\right)^3$$

$$= (0.26 \times 10^{45}/m^3)\left(\frac{R_i}{R}\right)^3$$

$$\frac{dn}{dt} = -(0.26 \times 10^{45}/m^3)\cdot 3\,\frac{R_i^3}{R^4}\,\frac{dR}{dt}$$

when $R = 10\,m$ this is

$$\frac{dn}{dt} \cong -\left(10^{41}/m^4\right) \cdot \frac{dR}{dt}$$

The density is replenished by annihilation processes at the rate

$$\frac{dn}{dt} \sim n(e^+) \cdot n(e^-) \cdot \sigma \cdot c$$

$$\sim \left(2\times10^{41}/m^3\right)^2 \cdot \left(10^{-28}\,m^2\right) \cdot \left(3\times10^{8}\,m/sec\right)$$

$$\sim 10^{62}/m^3 sec$$

this is larger than what we need to keep $n(\gamma)$ in equilibrium if

$$\frac{dR}{dt} \lesssim 10^{21}\,m/sec.$$

Of course, the real requirement is that $\frac{dR}{dt} \ll c$, to avoid shock waves and other processes that give local heating.

e.) With estimates similar to the above, the mean free path of a photon is given by

$$\ell^{-1} \sim n(\gamma) \cdot \sigma$$

$$\sim \left(2.6\times10^{41}/m^3\right)\left(10^{-28}\,m^2\right)$$

$$\sim \left(3\times10^{-12}\,m\right)^{-1}$$

or $\quad \ell \sim 3000 \text{ fm}$

quite a small distance.

The rest of this part is tricky and not sufficiently well explained in the exam. To get the right units, we need

$$P \sim \frac{\text{kg m/sec}^2}{\text{m}^2} \qquad \rho \sim \text{kg/m}^3$$

so that $\quad c \sim \text{m/sec.}$ Then $\rho$ must be the mass density of the relativistic gas. To compute this, realize that a cloud of relativistic particles that is collecting at rest has mass

$$m = E/c^2$$

then

$$P = \frac{11}{4}\frac{\pi^2}{45}\frac{T^4}{(\hbar c)^3}$$

$$\rho = \frac{E}{c^2} = \frac{11}{4}\frac{\pi^2}{15}\frac{T^4}{(\hbar c)^3}\frac{1}{c^2}$$

so

$$\frac{P}{\rho} = \frac{c^2}{3}$$

throughout the adiabatic expansion. Then also

$$\frac{\partial P}{\partial \rho} = \frac{c^2}{3}$$

so the speed of sound in the relativistic fluid is

$$c_s = \frac{c}{\sqrt{3}}$$

This is a very general conclusion. Here is an interesting application: In $He^4$ at finite temperature $T < T_\lambda$, we have a gas of phonons. These phonons travel at the speed $c_{s,ph}$. They interact strongly enough that the photon gas supports sound waves (called second sound) The excitations of second sound travel at the speed

$$c_{s,2} = \frac{c_{s,ph}}{\sqrt{3}}$$

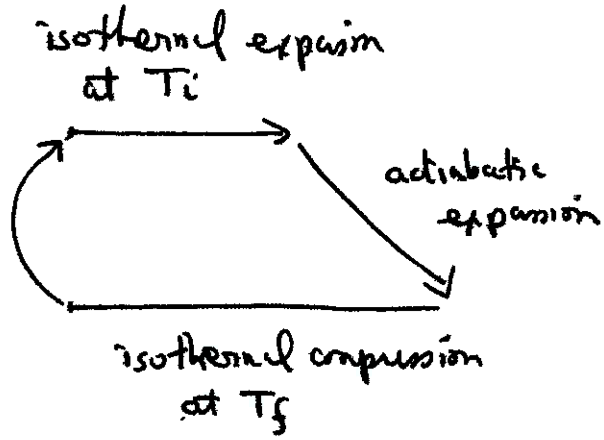
f.) The compression at $T_f = 20$ MeV follows the same formulae as in part (b). The work done on the balloon is

$$W = \frac{11}{4} \frac{\pi^2}{45} \frac{T_f^4}{(\hbar c)^3} \frac{4\pi}{3} R_f^3$$

The heat released by the balloon is

$$E_{rel.} = \frac{11}{45} \pi^2 \frac{T_f^4}{(\hbar c)^3} \frac{4\pi}{3} R_f^3$$

g.) Now we have a Carnot cycle




on the fourth leg of the cycle, the balloon has zero size. It contains no photons, electrons, or positrons, so there is no exchange of energy or entropy.

h.) In the stages of the cycle, we have, with $T_f^4 R_f^3 = T_i^4 R_i^3 \cdot \left(\frac{R_i}{R_f}\right.$

| | Heat absorbed | Work done |
|---|---|---|
| isothermal expansion | $\frac{11}{45}\pi^2 \frac{T_i^4}{(\hbar c)^3} \cdot \frac{4}{3}\pi R_i^3$ | $\frac{11}{4}\frac{\pi^2}{45}\frac{T_i^4}{(\hbar c)^3}\,\frac{4}{3}\pi R_i^3$ |
| adiabatic expansion | $0$ | $\frac{11}{4}\,\frac{\pi^2}{15}\,\frac{T_i^4}{(\hbar c)^3}\,\frac{4}{3}\pi R_i^3 \left(1-\frac{R_i}{R_f}\right)$ |
| isothermal compression | $-\frac{11}{45}\pi^2 \frac{T_i^4}{(\hbar c)^3} \cdot \frac{4\pi}{3} R_i^3 \cdot \left(\frac{R_i}{R_f}\right)$ | $-\frac{11}{4}\,\frac{\pi^2}{45}\,\frac{T_i^4}{(\hbar c)^3}\,\frac{4\pi}{3} R_i^3\, \frac{R_i}{R_f}$ |
| 4th leg | $0$ | $0$ |

The net work done is

$$\frac{11}{45}\pi^2 \frac{T_i^4}{(\hbar c)^3} \frac{4\pi}{3} R_i^3 \cdot \left(1 - \frac{R_i}{R_f}\right)$$

which also equals the net heat input. The efficiency is

$$\epsilon = \frac{\text{Net work}}{\text{Heat in at } T_i} = 1 - \frac{R_i}{R_f} = \left(1 - \frac{T_f}{T_i}\right)$$

which is exactly the expectation from Carnot's general analysis.

2.) a.) The picture of the atomic configuration is:

— A — B — A — B — A —

For $K=0$,

$$H = +J \sum_{i=1}^{N} \left( S_{Ai} S_{B(i+1)} + S_{Ai} S_{Bi} \right)$$

with $S_{Ai}, S_{Bi} \in \{\pm 1\}$

We can compute the free energy of this model by recognizing that it is equivalent to an Ising antiferromagnet

$$\mathcal{H} = +J \sum_{i=1}^{2N} s_i s_{i+1}$$

and using the transfer matrix method presented in class with periodic boundary conditions:

$$Z = \mathrm{tr}\, T^{2N}$$

with

$$T = \begin{pmatrix} e^{-\beta J} & e^{\beta J} \\ e^{\beta J} & e^{-\beta J} \end{pmatrix} \text{ acting on } \begin{pmatrix} s=\uparrow \\ s=\downarrow \end{pmatrix}$$

then $$Z = \lambda_>^{2N}$$ where $\lambda_>$ is the largest eigenvalue of T. This is

$$\lambda_> = (e^{\beta J} + e^{-\beta J}) \quad \text{for} \quad v_> = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

so

$$Z = (2\cosh\beta J)^{2N}$$

$$F = -2TN \log(2\cosh\beta J)$$

For $J=0$, the B atoms are not coupled to A's or to one another. Then

$$\sum_{S_{iB}} = 2^N$$

The A atoms can be treated by a transfer matrix, so

$$Z = 2^N \operatorname{tr} T^N$$

with $$T = \begin{pmatrix} e^{-\beta K} & e^{\beta K} \\ e^{\beta K} & e^{-\beta K} \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} S_A = \uparrow \\ S_A = \downarrow \end{pmatrix}$$

The largest eigenvalue of T is $\lambda_> = e^{\beta K} + e^{-\beta K} = 2\cosh\beta K$

so $$Z = (4\cosh\beta K)^N$$, and

$$F = -NT\log(4\cosh\beta K)$$

In both cases

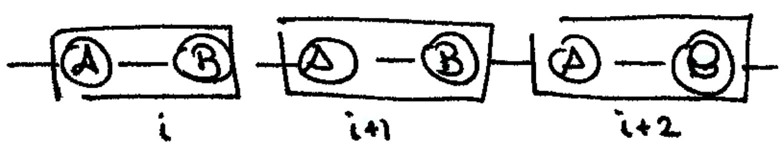
$$\langle S_{Ai}\rangle = \langle v_>|\, \underset{\sim}{S_A}\, |v_>\rangle$$

with $\quad v_> = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \underset{\sim}{S_A} = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$

$$= \frac{1}{2}\,(1\;1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 0$$

so $\quad \langle S_{Ai}\rangle = 0$

b.) When both J and K are nonzero, a more sophisticated approach is needed. Divide the chain as




and write a transfer matrix that acts on a 4-component vector

$$\begin{pmatrix} S_{Ai}=\uparrow & S_{Bi}=\uparrow \\ S_{Ai}=\uparrow & S_{Bi}=\downarrow \\ S_{Ai}=\downarrow & S_{Bi}=\uparrow \\ S_{Ai}=\downarrow & S_{Bi}=\downarrow \end{pmatrix}$$

The J and K interactions couple only neighboring "sites",

$i$ to $(i-1)$ and $i$ to $(i+1)$. To work out $T$ explicitly in this basis, consider the cases $J=0$ and $K=0$ separately.

For $J=0$

$$T = e^{-\beta K s_{Ai} s_{Ai+1}}$$

independent of the values of $s_{Bi}, s_{Bi+1}$. Then

$$T = \left(\begin{array}{cc|cc} e^{-\beta K} & e^{-\beta K} & e^{\beta K} & e^{\beta K} \\ e^{-\beta K} & e^{-\beta K} & e^{\beta K} & e^{\beta K} \\ \hline e^{\beta K} & e^{\beta K} & e^{-\beta K} & e^{-\beta K} \\ e^{\beta K} & e^{\beta K} & e^{-\beta K} & e^{-\beta K} \end{array}\right)$$

The largest eigenvalue of this matrix is

$$\lambda_> = 4\cosh\beta K \qquad v_> = \frac{1}{\sqrt{4}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

giving $F = -NT \log 4\cosh\beta K$ in accord with our previous result.

For $K=0$

$$T = e^{-\beta J s_{Ai} s_{Bi}}\, e^{-\beta J s_{Bi} s_{A\,i+1}}$$

$$= \left(\begin{array}{cc|cc} e^{-\beta J} & & & \\ & e^{+\beta J} & & \\ \hline & & e^{+\beta J} & \\ & & & e^{-\beta J} \end{array}\right) \left(\begin{array}{cc|cc} e^{-\beta J} & e^{+\beta J} & e^{-\beta J} & e^{\beta J} \\ e^{-\beta J} & e^{+\beta J} & e^{-\beta J} & e^{\beta J} \\ \hline e^{\beta J} & e^{-\beta J} & e^{\beta J} & e^{-\beta J} \\ e^{\beta J} & e^{-\beta J} & e^{\beta J} & e^{-\beta J} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} e^{-2\beta J} & 1 & e^{-2\beta J} & 1 \\ 1 & e^{2\beta J} & 1 & e^{2\beta J} \\ \hline e^{2\beta J} & 1 & e^{2\beta J} & 1 \\ 1 & e^{-2\beta J} & 1 & e^{-2\beta J} \end{array}\right)$$

Look for an eigenvector that is symmetric under $S \to -S$

$$v = \frac{1}{\sqrt{2(a^2+b^2)}} \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix}$$

$$T v = \frac{1}{\sqrt{2(a^2+b^2)}} \begin{pmatrix} (e^{-2\beta J}+1)(a+b) \\ (e^{2\beta J}+1)(a+b) \\ (e^{2\beta J}+1)(a+b) \\ (e^{-2\beta J}+1)(a+b) \end{pmatrix}$$

so
$$\frac{a}{b} = \frac{1+e^{-2\beta J}}{1+e^{2\beta J}} = e^{-2\beta J}$$

$$v = \frac{1}{\sqrt{*}} \begin{pmatrix} e^{-2\beta J} \\ 1 \\ 1 \\ e^{-2\beta J} \end{pmatrix} \qquad \lambda_> = (e^{2\beta J}+1)(1+e^{-2\beta J}) = (2\cosh \beta J)^2$$

so indeed
$$F = -2NT \log(2\cosh\beta J)$$

as above.

Now, combine these. The full transfer matrix is

$$T' = \left(\begin{array}{cc|cc} e^{-\beta K-2\beta J} & e^{-\beta K} & e^{\beta K-2\beta J} & e^{\beta K} \\ e^{-\beta K} & e^{-\beta K+2\beta J} & e^{\beta K} & e^{\beta K+2\beta J} \\ \hline e^{\beta K+2\beta J} & e^{\beta K} & e^{-\beta K+2\beta J} & e^{-\beta K} \\ e^{\beta K} & e^{\beta K-2\beta J} & e^{-\beta K} & e^{-\beta K-2\beta J} \end{array}\right)$$

again, look for an eigenvector of the form

$$v = \left(\begin{array}{c} a \\ b \\ \hline b \\ a \end{array}\right)$$

$$Tv = \begin{pmatrix} a\left(e^{\beta K} + e^{-\beta K-2\beta J}\right) + b\left(e^{\beta K-2\beta J} + e^{-\beta K}\right) \\ a\left(e^{-\beta K} + e^{\beta K+2\beta J}\right) + b\left(e^{-\beta K+2\beta J} + e^{\beta K}\right) \\ a\left(e^{-\beta K} + e^{\beta K+2\beta J}\right) + b\left(e^{-\beta K+2\beta J} + e^{\beta K}\right) \\ a\left(e^{\beta K} + e^{-\beta K-2\beta J}\right) + b\left(e^{\beta K-2\beta J} + e^{-\beta K}\right) \end{pmatrix}$$

so this is consistent and

$$\lambda_s a = a\left(e^{\beta K} + e^{-\beta K} e^{-2\beta J}\right) + b\left(e^{-\beta K} + e^{\beta K-2\beta J}\right)$$

$$\lambda_s b = a\left(e^{-\beta K} + e^{\beta K} e^{+2\beta J}\right) + b\left(e^{\beta K} + e^{-\beta K+2\beta J}\right)$$

$$\frac{a}{b} = \frac{e^{-\beta K} + e^{\beta K - 2\beta J}}{(\lambda - (e^{\beta K} + e^{-\beta K} e^{-2\beta J}))} = \frac{(\lambda - (e^{\beta K} + e^{-\beta K} e^{2\beta J}))}{(e^{-\beta K} + e^{\beta K} e^{2\beta J})}$$

$$\lambda^2 - \left[2e^{\beta K} + e^{-\beta K}(e^{2\beta J} + e^{-2\beta J})\right]\lambda + (e^{-2\beta K} + e^{2\beta K} + e^{2\beta J} + e^{-2\beta J})$$

$$= e^{-2\beta K} + e^{2\beta K} + e^{2\beta J} + e^{-2\beta J}$$

$$\lambda\,(\lambda - 2\,(e^{\beta K} + e^{-\beta K}\cosh 2\beta J)) = 0$$

The largest eigenvalue is

$$\lambda = 2\,(e^{\beta K} + e^{-\beta K}\cosh 2\beta J)$$

then

$$F = -NT \log\left[2\,(e^{\beta K} + e^{-\beta K}\cosh 2\beta J)\right]$$

since $J \Rightarrow 0 \quad \lambda \rightarrow 4\cosh \beta K$

$K \rightarrow 0 \quad \lambda \rightarrow 2\,(1 + \cosh 2\beta J) = 4\cosh^2 \beta J$

this does reproduce both limits above.

Here is a nicer way to get this result: The sum over $S_{Bi}$ has the form

$$A - B - A$$

$$\sum_{S_{Bi}} e^{-\beta J S_{Ai} S_{Bi}} \, e^{-\beta J S_{Ai+1} S_{Bi}}$$

$$= \sum_{S_{Bi}=\pm 1} \left(\cosh\beta J - S_{Ai} S_{Bi} \sinh\beta J\right)\left(\cosh\beta J - S_{Bi} S_{Ai+1} \sinh\beta J\right)$$

$$= 2\left(\cosh^2\beta J + S_{Ai} S_{Ai+1} \sinh^2\beta J\right)$$

Then we can write the transfer matrix between $S_A$ spins only

as

$$T = \begin{pmatrix} 2 e^{-\beta K} \cosh 2\beta J & 2 e^{\beta K} \\ 2 e^{\beta K} & 2 e^{-\beta K} \cosh 2\beta J \end{pmatrix}$$

acting on

$$\begin{pmatrix} S_A = \uparrow \\ S_A = \downarrow \end{pmatrix}$$

The leading eigenvalue is

$$\lambda_> = 2\left(e^{\beta K} + e^{-\beta K} \cosh 2\beta J\right)$$

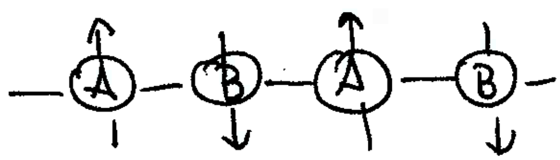
for $v_> = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

the second eigenvalue is

$$\lambda_< = 2\left[e^{-\beta K}\cosh 2\beta J - e^{\beta K}\right]$$

for $\quad v_< = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

c.) At $T=0$, the system is described by the lowest energy states of the Hamiltonian. One candidate for such a state is

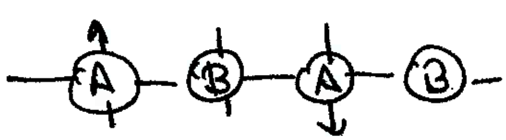


$S_{Ai} = +1 \quad S_{Bi} = -1$

(or $\quad S_{Ai} = -1 \quad S_{Bi} = +1$)

For this state $\quad \mathcal{H} = -2NJ + NK$

Another candidate is



$S_{Ai} = (-1)^i$

For this state, for any configuration of $S_{Bi}$,

$$\mathcal{H} = -NK$$

then at $T=0$

if $J>K$ the $S_{Ai}$ are ferromagnetically ordered

$J<K$ the $S_{Ai}$ are antiferromagnetically ordered

d.) It is very tricky to compute the correlation function of the $S_{Ai}$ using the 4x4 transfer matrix. However, it is easy to do this computation using the 2x2 transfer matrix on p. 18. We can directly apply the analysis done in class to find

$$\langle S_{Ai}\, S_{A(i+n)} \rangle = \left(\frac{\lambda_<}{\lambda_>}\right)^n$$

or

$$\langle S_{Ai}\, S_{A(i+n)} \rangle = \left[\frac{e^{-\beta K}\cosh 2\beta J - e^{\beta K}}{e^{-\beta K}\cosh 2\beta J + e^{\beta K}}\right]^n$$

for $J \gg K$, or, more specifically, if

$$\cosh 2\beta J > e^{2\beta K}$$

the quantity in brackets is positive and we have ferromagnetic spin correlations. If

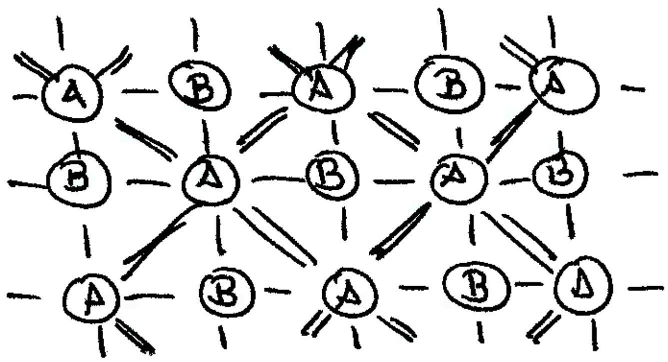
$$\cosh 2\beta J < e^{2\beta K}$$

the quantity in brackets is negative,

$$\langle S_{Ai}\, S_{A(i+n)} \rangle \sim a^n \cdot (-1)^n \qquad \underline{a > 0}$$

and we have antiferromagnetic spin correlations.

e.) Now consider a 2-d system:




To do mean field theory with ferromagnetic order of the A's, write:

$$\langle S_{Ai} \rangle = S_A \qquad \langle S_{Bi} \rangle = S_B$$

The statistical weight of an A spin will be

$$e^{-4\beta J (S_{Ai}) \cdot S_B} \; e^{-4\beta K (S_{Ai}) \cdot S_A}$$

similarly, the statistical weight of a B spin will be

$$e^{-4\beta J(S_{Bi})\cdot S_A}$$

The self-consistency equations are then

$$S_A = \frac{e^{-4\beta J S_B - 4\beta K S_A} - e^{4\beta J S_B + 4\beta K S_A}}{e^{-4\beta J S_B - 4\beta K S_A} + e^{4\beta J S_B + 4\beta K S_A}}$$

or

$$S_A = -\tanh(4\beta K S_A + 4\beta J S_B)$$

and similarly

$$S_B = -\tanh(4\beta J S_A)$$

f.) For small $S_A, S_B$, we can linearize these equations and find

$$S_A = -(4\beta K S_A + 4\beta J S_B) + \cdots$$

$$S_B = -4\beta J S_A + \cdots$$

eliminating $S_B$

$$S_A = +(4\beta J)^2 S_A - 4\beta K S_A + \cdots$$

The critical temperature occurs when the two sides of the equation are tangent

$$1 = (4\beta_c J)^2 - 4\beta_c K$$

then $$\beta_c^2 - \frac{1}{4J^2}\beta_c K = \frac{1}{16J^2}$$

$$\left(\beta_c - \frac{K}{8J^2}\right)^2 = \frac{1}{16J^2} + \frac{K^2}{64J^4}$$

$$\beta_c = \frac{1}{8}\frac{K}{J^2} + \frac{1}{4J}\left(1 + \frac{1}{4}\frac{K^2}{J^2}\right)^{1/2}$$

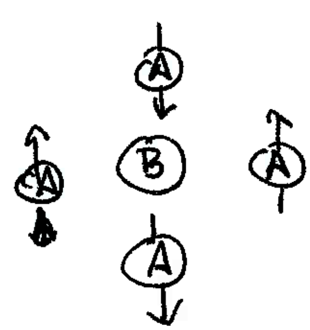
or $$T_c^{(\text{ferro})} = 4J\left[\left(1 + \frac{1}{4}\frac{K^2}{J^2}\right)^{1/2} + \frac{1}{2}\frac{K}{J}\right]^{-1}$$

for $K \ll J$, $\quad T_c^{(\text{ferro})} = 4J - 2K + \ldots$

g.) If the A atoms have antiferromagnetic order

$$\langle s_{Ai} \rangle = (-1)^i s_A$$

there is no ordering force on the B spins in mean field theory



The statistical weight for a B spin is

$$e^{-2\beta J s_A s_{Bi} + 2\beta J s_A s_{Bi}} = 1$$

then self-consisting gives $\langle s_{Bi} \rangle = 0$. The $s_{Ai}$ weight is then

$$e^{+4\beta K s_A s_{Ai}}$$

giving

$$s_A = \tanh 4\beta K s_A$$

and

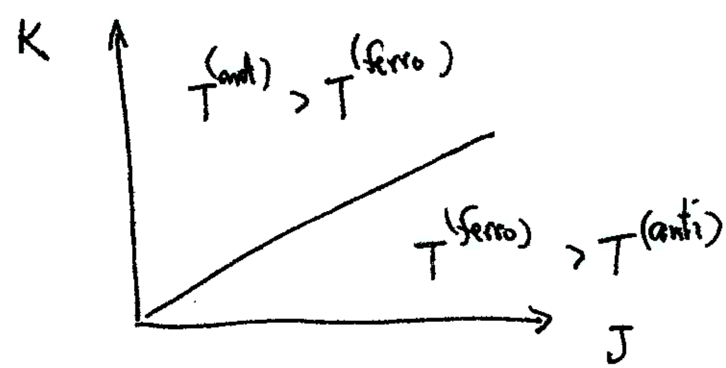
$$T_c^{(anti)} = 4K$$

The two values of $T_c$ are surfaces over the $(J,K)$ plane. The surfaces cross when $T_c^{(ferro)} = T_c^{(anti)}$

$$\left(\frac{1}{4k}\right)^2 - \frac{1}{4J^2}\frac{1}{4k}K = \frac{1}{16J^2}$$

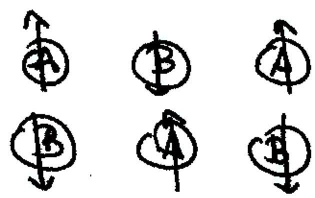
$$\frac{1}{16K^2} = \frac{1}{8J^2}$$

$$K = \frac{1}{\sqrt{2}} J$$



h) At $T=0$, we are concerned with the ground states of the model. The candidate ground states are:

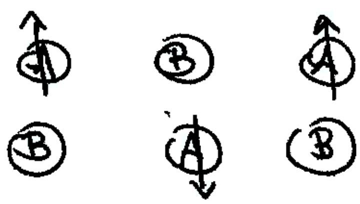
ferro:



or

$$\mathcal{H} = -4JN + 2KN$$

antiferro



$$\mathcal{H} = -2KN$$ for any alignment of the B's.

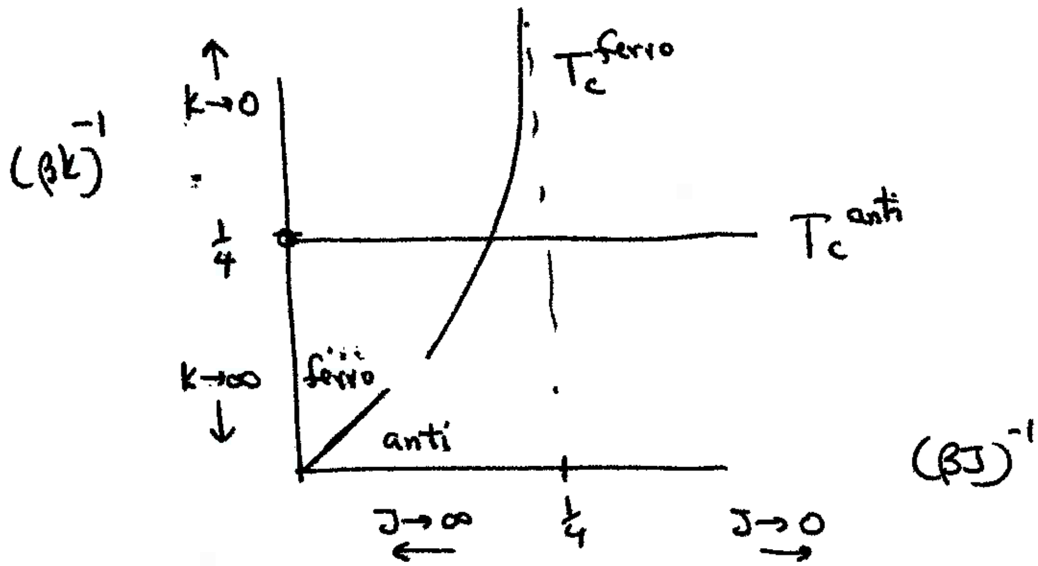
so at $T=0$

$J > K$ ferro-order of A's

$J < K$ antiferro-order of A's.

i) Now we can sketch a phase diagram consistent with this data. The physics depends only on the combinations $(\beta J)$ and $(\beta K)$. I will give a sketch in the plane of the temperature-like variables $(\beta J)^{-1}$ and $(\beta K)^{-1}$

First, plot the two values of $T_c$ and the boundary at low temperatures




We can join these up as follows: