Physics 212 - Final Exam
Solutions

1.) a.) The pressure on the membrane comes from two sources:

① the osmotic pressure due to the sugar molecules

This is
$$\frac{N}{V}T = n T$$

the net downward osmotic pressure is

$$n(1-f)T$$

② the pressure due to the weight of the water in the column.

the net pressure is $\rho g h$

Transfer water to increase the column height by dh.

The new pressure is

$$P + dP = n\left(\frac{H+h}{H+h+dh}\right)T - fnT + \rho g(h+dh)$$

so

$$dP = -n \frac{dh}{H+h} T + \rho g \, dh$$

and is minimized when

$$\frac{nT}{H+h} = \rho g$$

It is odd but correct that this formula allows $h < 0$ as long as $H + h > 0$. For $h > 0$ we must have

$$nT > \rho g H$$

Now put in numbers:

$$\rho g = 1 \, g/cm^3 \cdot 980 \, cm/sec^2 = 980 \, dyne/cm^3$$

for 1 mole/l $\quad n = 10^{-3} N_A / cm^3 = 6.02 \times 10^{20} / cm^3$

$$k_B T = 1.38 \times 10^{-16} erg/°K \cdot 300°K$$

$$n k_B T = 2.5 \times 10^{7} erg/cm^3 = dyne/cm^2$$

so $H + h = 25000 cm = 250 m$!

This is actually correct. A 10^{-3} mole/l sugar solution would be closer to your experience.

b.) The answer is the difference in chemical potential of the sugar on the two sides of the membrane.

$$\mu = T \log \frac{N}{V} + const$$

so

$$\Delta\mu = T\log n - T\log nf$$

$$= -T\log f$$

c.) Let A be the area of the cylinder.

$$N = nA(H+h)$$

We can raise h if n is increased:

$$0 = d\left(\frac{nT}{H+h} - \rho g\right) = \frac{dn\,T}{H+h} - \frac{nT\,dh}{(H+h)^2}$$

$$dn = \frac{n\,dh}{(H+h)}$$

$$dN = dn\,A(H+h) + nA\,dh$$

$$= nA\,dh + nA\,dh$$

$$= 2nA\,dh$$

The free energy needed is then

$$dF = 2nTA\,dh \cdot \left(\log\frac{1}{f}\right)$$

$$= 2A[\rho g(H+h)]\log\frac{1}{f}\cdot dh$$

It is odd that this can → 0 when f → 1. But remember that I have assumed an infinite reservoir that is replenished as the sugar is transported.

2.) a.) For neutron matter

$$n = \frac{N}{V} = 2 \cdot \int^{k_F} \frac{d^3k}{(2\pi)^3} = \frac{2 \cdot 4\pi}{8\pi^3} \int_0^{k_F} dk\, k^2$$

$$= \frac{1}{3\pi^2} k_F^3$$

$$\frac{E}{V} = 2 \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{2m_n} = \frac{1}{\pi^2} \int_0^{k_F} dk \frac{k^4}{2m_n}$$

$$= \frac{1}{10\pi^2 m_n} k_F^5 = \frac{3}{5} \frac{N}{V} \varepsilon_F$$

$$P = -\frac{\partial}{\partial V} E = -\frac{\partial}{\partial V} \left(\frac{E}{V} \cdot V \right)$$

now $\frac{E}{V} \sim k_F^5 \sim \left(\frac{N}{V}\right)^{5/3}$ so $\frac{E}{V} \cdot V \sim \frac{1}{V^{2/3}}$

$$P = \frac{2}{3} \frac{E}{V} = \frac{1}{15\pi^2 m_n} k_F^5$$

where $k_F = (3\pi^2 n)^{1/3}$

for quark matter

$d \leftrightarrow u \leftrightarrow s$ freely, so the chemical potentials are equal.

at $T=0$ $\mu_q = \varepsilon_F$.

$$n = \underset{u,d,s}{3} \cdot \underset{\text{spin }\frac{1}{2}}{2} \cdot \underset{\text{colors}}{3} \cdot \int^{k_F} \frac{d^3k}{(2\pi)^3}$$

$$= \frac{9}{3\pi^2} k_F^3 = \frac{3}{\pi^2} k_F^3$$

$$\frac{E}{V} = 9 \cdot 2 \cdot \int^{k_F} \frac{d^3k}{(2\pi)^3}\, ck = \frac{9}{\pi^2} \int^{k_F} dk\, k^3 \cdot c$$

$$= \frac{9}{4\pi^2} c k_F^4$$

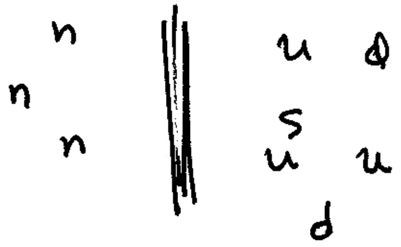
$$P = -\frac{\partial}{\partial V} E = \frac{1}{3} \frac{E}{V}$$

$$P = \frac{3}{4\pi^2} c k_F^4 \quad \text{where } k_F = \left(\frac{\pi^2}{3} n\right)^{1/3}$$

we must add to this the pressure due to the volume energy $\Delta P = -\frac{\partial}{\partial V}(BV) = -B$

$$P = \frac{3}{4\pi^2} c k_F^4 - B$$

If we have an interface between neutron matter and quark matter



The pressures are equal. $P(n) = P(q)$

The reaction $n \rightleftarrows u+d+d$ is in equilibrium

so $\mu_n = 3\mu_q$

at $T=0$ $\varepsilon_F(n) = 3\varepsilon_F(q)$

$$m_n c^2 + \frac{k_F^2(n)}{2m_n} = 3ck_F(q)$$

or $$k_F(q) = \frac{1}{3} m_n c \left[1+\frac{k_F^2(n)}{2(m_n c)^2}\right]$$

the equality of pressures is

$$\frac{k_F^5(n)}{15\pi^2 m_n} = \frac{3}{4\pi^2} c\, k_F^4(q) - B$$

$$\frac{k_F^5(n)}{15\pi^2 m_n} = \frac{1}{108\pi^2} m_n^4 c^5 \left(1+\frac{k_F^2(n)}{2(m_n c)^2}\right)^4 - B$$

$$k_F^5(n) = \frac{5}{36}(m_n c)^5 \left(1+\frac{k_F^2(n)}{2(m_n c)^2}\right)^4 - 15\pi^2 m_n B$$

Now I'll make an approximation: neutron matter stay nonrelativistic. Then

$$\frac{k_F^2(n)}{(m_n c)^2} \ll 1$$

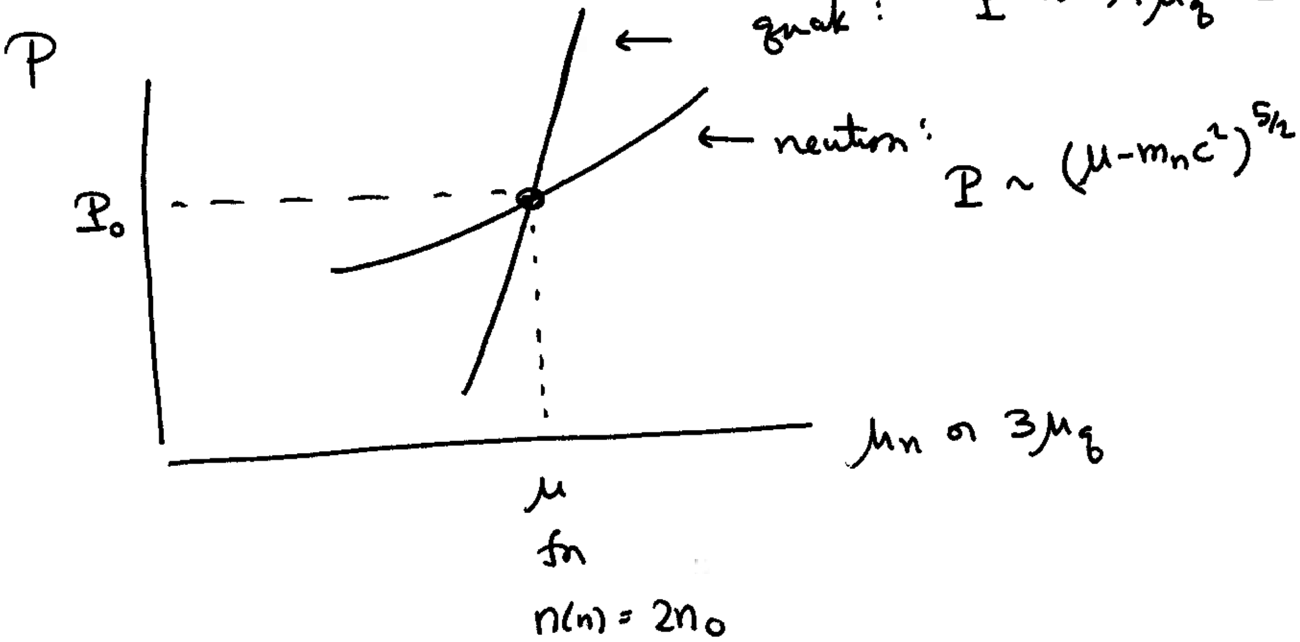
$$k_F(n) \approx \left[\frac{5}{36}(m_n c)^5 - 15\pi^2 m_n B \right]^{\frac{1}{5}}$$

We could solve this for B using the $k_F(n) = (3\pi^2 \cdot 2n_0)^{\frac{1}{3}}$

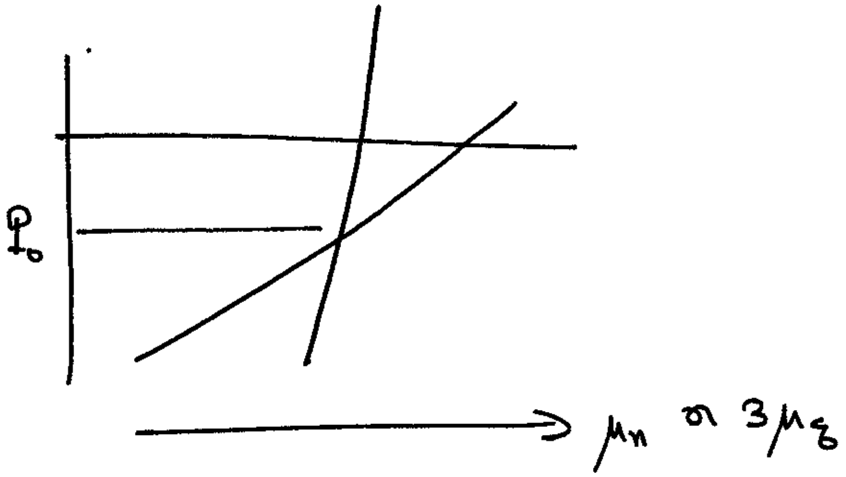
But anyway $$k_F(q) \approx \frac{1}{3} m_n c$$

so $$n(q) \approx \frac{1}{27\pi^2}(m_n c)^3$$

b.) At T=0, as a function of ε_F or μ, the pressures of the two phases behave as.



At pressures higher than P_0



we no longer have coexistence, and $\mu_q < \frac{1}{3}\mu_n$

So the quark matter phase has lower free energy, and the whole system turns into quark matter.

(c.) The formula for the pressure of a free Fermi gas at finite temperature is

$$P = -\frac{\Phi}{V} = \frac{1}{\beta}\, g \int \frac{d^3k}{(2\pi)^3} \log\left(1+e^{-\beta(\varepsilon-\mu)}\right)$$

$$= \frac{1}{\beta}\,\frac{g}{2\pi^2} \int dk\, k^2 \log\left(1+e^{-\beta(\varepsilon-\mu)}\right)$$

For neutrons $\quad g = 2 \qquad \varepsilon(k) = m_n c^2 + \frac{k^2}{2m_n}$

$$dk\, k^2 = \left(\frac{2m_n}{2}\right)^{3/2} d\bar{\varepsilon}\; \bar{\varepsilon}^{1/2}$$

$$\bar{\varepsilon} = \varepsilon(k) - m_n c^2$$

now integrate by parts

$$P_n = \frac{1}{\pi^2}\frac{(2m_n)^{3/2}}{3}\int d\bar{\varepsilon}\;\bar{\varepsilon}^{3/2}\;\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

For quarks $\quad g=18 \qquad \varepsilon(k) = ck$

$$dk\;k^2 = \frac{1}{c^3}\,d\varepsilon\;\varepsilon^2$$

now integrate by parts

$$P_q = \frac{3}{\pi^2}\frac{1}{c^3}\int d\varepsilon\;\varepsilon^3\;\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

In class, we discussed how to get low-T approximations to these formulae:

$$\int_0^\infty d\varepsilon\;H(\varepsilon)\;\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \cong \int_0^\mu d\varepsilon\;H(\varepsilon) + \frac{\pi^2}{6}T^2 H'(\mu) + \cdots$$

so

$$\int d\bar{\varepsilon}\;\bar{\varepsilon}^{3/2}\;\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \cong \frac{2}{5}(\mu-m_nc^2)^{5/2} + \frac{\pi^2}{4}T^2(\mu-m_nc^2)^{1/2} + \cdots$$

$$\int d\varepsilon\;\varepsilon^3\,\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \cong \frac{1}{3}\mu^4 + \frac{\pi^2}{2}T^2\mu^2 + \cdots$$

so

neutron matter:

$$n = \frac{2\sqrt{2}\, m^{3/2}}{3\pi^2}(\mu - m_n c^2)^{3/2} + \sqrt{2}\,\frac{m^{3/2}}{12}T^2\frac{1}{(\mu - m_n c^2)^{1/2}} + \cdots$$

$$P = \frac{2\,(2m_n)^{3/2}}{15\pi^2}(\mu - m_n c^2)^{5/2} + \frac{(2m_n)^{3/2}}{12}T^2(\mu - m_n c^2)^{1/2} + \cdots$$

quark matter:

$$n = \frac{3}{\pi^2}\frac{\mu^3}{c^3} + 3T^2\frac{\mu}{c^3} + \cdots$$

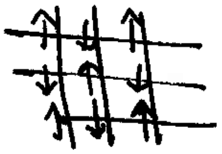
$$P = \frac{1}{\pi^2}\frac{\mu^4}{c^3} + \frac{3}{2}T^2\frac{\mu^2}{c^3} + \cdots$$
$$- B$$

now set $P_n = P_q$ $\mu_n = 3\mu_q$ and use the above value of B.

This is *very* messy! Anyone who got this far gets full credit!

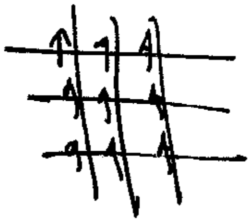
3.) a.) At $T=0$ candidate ground states are

complete antiferromagnetism



$E = -dJN \qquad \langle s\rangle = 0 \qquad \langle t\rangle = 1$ or -1

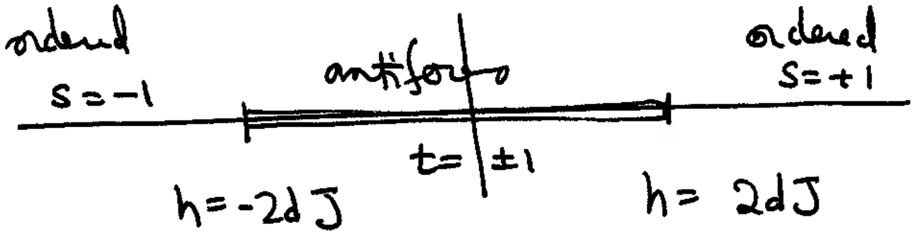
complete magnetic order



$E = +dJN - hN$

$\langle s\rangle = 1$ or $-1 \qquad \langle t\rangle = 0$

so



b.) Let $s_e = \langle s_i \rangle$ on even sites

$s_o = \langle s_i \rangle$ on odd sites

$$s = \tfrac{1}{2}(s_e + s_o) \qquad t = \tfrac{1}{2}(s_e - s_o)$$

or $$s_e = s + t \qquad s_o = s - t$$

the effective single-site Hamiltonian for <u>even</u> sites is

$$e^{-2dJ\beta\, s_o s_i} \; e^{\beta h s_i}$$

so

$$\begin{aligned}\langle s_e \rangle &= \tanh(-2d\beta J s_o + \beta h) \\ &= \tanh(2d\beta J(t-s) + \beta h)\end{aligned}$$

the effective single-site Hamiltonian for <u>odd</u> sites is

$$e^{-2d\beta J\, s_e s_i} \; e^{\beta h s_i}$$

so

$$\begin{aligned}\langle s_o \rangle &= \tanh(-2d\beta J s_e + \beta h) \\ &= \tanh(-2d\beta J(t+s) + \beta h)\end{aligned}$$

so

the self-consistency equations are:

$$s = \frac{1}{2}\left\{\tanh\left(2d\beta J(t-s) + \beta h\right) + \tanh\left(-2d\beta J(t+s) + \beta h\right)\right\}$$

$$t = \frac{1}{2}\left\{\tanh\left(2d\beta J(t-s) + \beta h\right) + \tanh\left(2d\beta J(t+s) - \beta h\right)\right\}$$

c.) Look first at $h=0$

$$s = \frac{1}{2}\left[\tanh 2d\beta J(t-s) - \tanh 2d\beta J(t+s)\right]$$
$$t = \frac{1}{2}\left[\tanh 2d\beta J(t-s) + \tanh 2d\beta J(t+s)\right]$$

$s=t=0$ is always a solution. Varying t at $s=0$

$$t = \tanh(2d\beta J\, t)$$

this is the mean field equation of the Ising model and has a critical point at

$$2d\beta_c J = 1$$

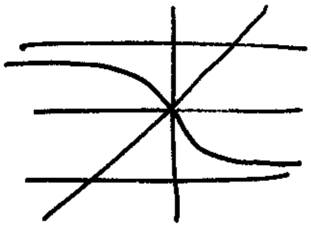
If $t =$ nontrivial solution of this equation

$s=0$

that solves both equations.

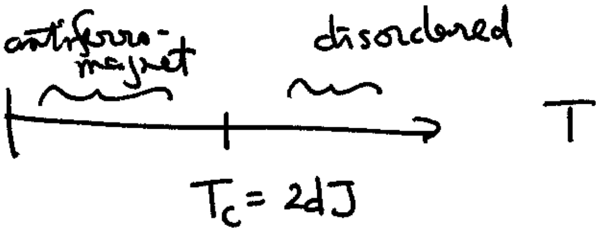
conversely, the problem for $t=0$, $s \neq 0$ is

$$s = -\tanh 2d\beta J s$$



which has no nontrivial self-consistent solution.

so we find



Now turn on a small h. The s equation becomes.

$$s = \frac{1}{2}\Big\{\big(2d\beta J(t-s) + \beta h\big) + \big(-2d\beta J(t+s) + \beta h\big)\Big\}$$

$$= -2d\beta J s + \beta h$$

or

$$s \cong \frac{\beta h}{1+2d\beta J} = s(h)$$

Put this into the t equation:

$$t = \frac{1}{2}\Big\{\tanh\Big(2d\beta J t + \beta h\Big(1 - \frac{2d\beta J}{1+2d\beta J}\Big)\Big) + \tanh\Big(2d\beta J t + (-\beta h)\Big(1 - \frac{2d\beta J}{1+2d\beta J}\Big)\Big)\Big\}$$

$$t = \frac{1}{2}\Big\{\tanh\Big(2d\beta J t + \Big(\frac{\beta h}{1+2d\beta J}\Big)\Big)$$
$$+\tanh\Big(2d\beta J t - \Big(\frac{\beta h}{1+2d\beta J}\Big)\Big)\Big\}$$

expand for small arguments:

$$t = 2d\beta J t$$
$$- \frac{1}{3}\cdot\frac{1}{2}\Big\{\Big(2d\beta J t + \frac{\beta h}{1+2d\beta J}\Big)^3 + \Big(2d\beta J t - \frac{\beta h}{1+2d\beta J}\Big)^3\Big\}$$
$$+ \cdots$$

$$= 2d\beta J t - \frac{1}{3}(2d\beta J t)^3$$
$$- \frac{1}{2}\, 2d\beta J t \cdot\Big(\frac{\beta h}{1+2d\beta J}\Big)^2 + \cdots$$

$$= \underbrace{2d\beta J\Big[1 - \frac{1}{2}\Big(\frac{\beta h}{1+2d\beta J}\Big)^2 + \cdots}\Big]\, t + O(t^3)$$

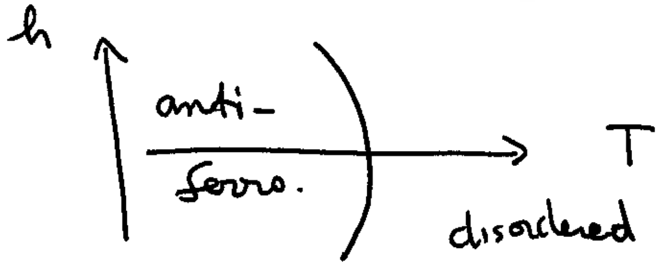
- the criterion for T_c is that this coefficient = 1

so

$$T_c = 2dJ\Big(1 - \frac{1}{2}\Big(\frac{h/2dJ}{2}\Big)^2 + \cdots\Big)$$

$$T_c = 2dJ\Big(1 - \frac{1}{32}\Big(\frac{h}{dJ}\Big)^2 + \cdots\Big)$$

this is still a <u>continuous</u> phase transition.



d.) <u>But</u> we saw in part (a) that the phase transition at $T=0$ is discontinuous.

So the phase diagram must be: