Physics 212 - Problem Set #8

Solutions

1.) $$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega_0^2 \theta = \frac{f(t)}{m}$$

a.) The AC susceptibility is the response function $\chi(t)$ such that

$$\theta(t) = \int dt' \; \chi(t-t') \, f(t')$$

Its Fourier transform satisfies

$$\theta(\omega) = \chi(\omega) \, f(\omega)$$

Now $$\theta(t) = \int \frac{d\omega}{2\pi} \, e^{-i\omega t} \, \theta(\omega)$$

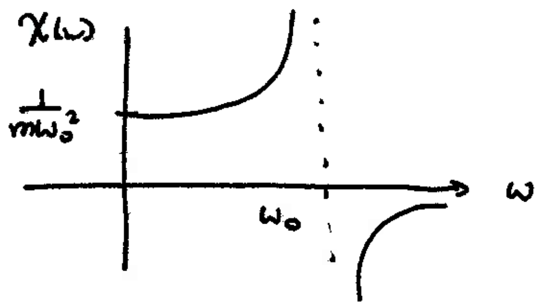
so the Fourier transform of $\frac{d\theta}{dt}$ is $-i\omega\,\theta(\omega)$, etc.

The transform of the differential equation is

$$[-\omega^2 - i\omega\gamma + \omega_0^2] \, \theta(\omega) = \frac{1}{m} f(\omega)$$

So $$\chi(\omega) = \frac{1}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}$$

For $\gamma = 0$, χ is real and has a singularity at $\omega = \omega_0$



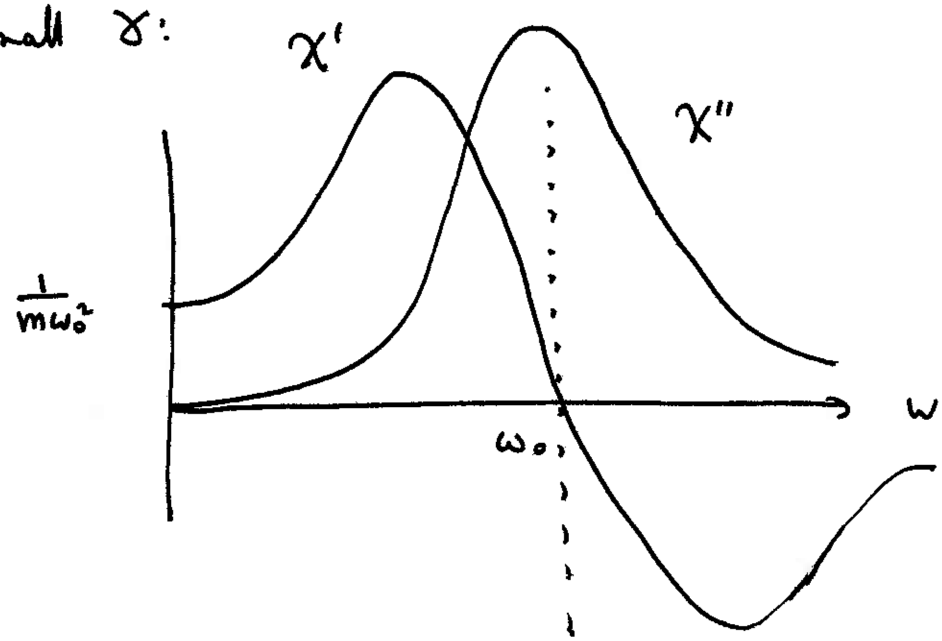
For $\gamma \neq 0$

$$\chi(\omega) = \frac{1}{m(\omega_0^2-\omega^2-i\omega\gamma)} \frac{(\omega_0^2-\omega^2+i\omega\gamma)}{(\omega_0^2-\omega^2+i\omega\gamma)}$$

$$= \frac{1}{m\left((\omega^2-\omega_0^2)^2+(\omega\gamma)^2\right)} \cdot (\omega_0^2-\omega^2+i\omega\gamma)$$

$$\chi'(\omega) = \frac{\omega_0^2-\omega^2}{m\left((\omega^2-\omega_0^2)^2+(\omega\gamma)^2\right)} \qquad \chi''(\omega) = +\frac{\omega\gamma}{m\left((\omega^2-\omega_0^2)^2+(\omega\gamma)^2\right)}$$

for small γ:



Note that $\omega\chi''(\omega) > 0$ as we expect.

b.) The poles of $\chi(\omega)$ are the solutions of

$$\omega_0^2 - \omega^2 - i\omega\gamma = 0$$

$$\omega_0^2 - \gamma^2/4 = (\omega + \tfrac{i}{2}\gamma)^2$$

$$\omega = -\frac{i}{2}\gamma \pm \left[\omega_0^2 - \gamma^2/4\right]^{\frac{1}{2}}$$

for $\omega_0^2 > \gamma^2/4$ both poles have $\operatorname{Im}\omega = -\gamma/2 < 0$

for $\omega_0^2 < \gamma^2/4$ (overdamped case)

$$\omega = -i\left[\frac{\gamma}{2} \pm \left[\left(\frac{\gamma}{2}\right)^2 - \omega_0^2\right]^{\frac{1}{2}}\right]$$

both poles are on the imaginary axis.

In either case, both poles are in the lower ½ plane.

c.) For $f(t) = A\cos\omega t = \operatorname{Re}\left[A e^{-i\omega t}\right]$

look for a solution to the differential equation of the form

$$\theta(t) = \operatorname{Re}\left[\Theta\, e^{-i\omega t}\right]$$

$$\Theta\,[-\omega^2 - i\omega\gamma + \omega_0^2] = A/m$$

$$\Theta = \frac{A}{m[\omega_0^2 - \omega^2 - i\omega\gamma]} = \frac{A\,(\omega_0^2 - \omega^2 + i\omega\gamma)}{m\,[(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2]}$$

then $\quad \theta = \mathrm{Re}\, \Theta e^{-i\omega t}$

$$= \frac{A}{m((\omega_0^2-\omega^2)^2 + (\omega\gamma)^2)}\,[(\omega_0^2-\omega^2)\cos\omega t + \omega\gamma \sin\omega t]$$

the power dissipation is

$$P = f \cdot \frac{d\theta}{dt} = A\cos\omega t \cdot \frac{d}{dt}\theta(t)$$

taking the average over a cycle:

$$\langle \cos\omega t \ \frac{d}{dt}\cos\omega t \rangle = 0$$

$$\langle \cos\omega t \ \frac{d}{dt}\sin\omega t \rangle = \omega \langle \cos^2\omega t \rangle = \frac{1}{2}\omega$$

so

$$\langle P \rangle = \frac{A^2\,\omega\gamma}{m\,[(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2]} \cdot \frac{\omega}{2}$$

$$= \frac{\omega}{2}\,|A|^2 \cdot \chi''(\omega)$$

in agreement with (10.37).

d.) Using the fluctuation - dissipation theorem

$$\chi''(\omega) = \frac{\beta\omega}{2} C(\omega)$$

we find

$$C(\omega) = \frac{2}{m\beta\omega} \frac{\omega\gamma}{[(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2]}$$

$$= \frac{2T \cdot \gamma}{m[(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2]}$$

To find $C(t)$ we invert this

$$C(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{2T\gamma/m}{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}$$

In particular,

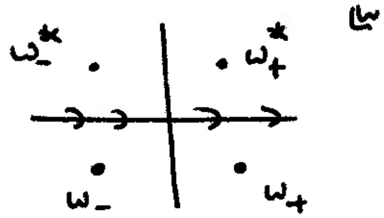
$$C(0) = \int \frac{d\omega}{2\pi} \frac{2T\gamma/m}{(\omega^2-\omega_0^2)^2 + (\omega\gamma)^2}$$

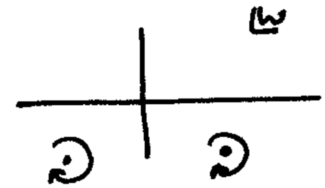
Let $\omega_\pm$ be the two solutions on p. 3

$$\omega^2 - \omega_0^2 + i\omega\gamma = (\omega - \omega_+)(\omega - \omega_-)$$

$$(\omega^2-\omega_0^2)^2 + (\omega\gamma)^2 = (\omega-\omega_+)(\omega-\omega_-)(\omega - \omega_+^*)(\omega-\omega_-^*)$$

ω_+ and ω_- are in the lower ½ plane. Do the integral as a contour integral, picking up these poles:





$$C(0) = (-2\pi i)\ \frac{2T\gamma/m}{2\pi} \left\{ \frac{1}{\omega_+ - \omega_-}\ \frac{1}{\omega_+ - \omega_+^*}\ \frac{1}{\omega_+ - \omega_-^*} + \frac{1}{\omega_- - \omega_+}\ \frac{1}{\omega_- - \omega_+^*}\ \frac{1}{\omega_- - \omega_-^*} \right\}$$

now $(\omega_+ - \omega_+^*)(\omega_+ - \omega_-^*) = \omega_+^2 - \omega_0^2 - i\omega_+\gamma = -2i\omega_+\gamma$

$\updownarrow$

$$(\omega_- - \omega_+^*)(\omega_- - \omega_-^*) = \omega_-^2 - \omega_0^2 - i\omega_-\gamma = -2i\omega_-\gamma$$

$$\omega_+ - \omega_- = 2\left[\omega_0^2 - \gamma^2/4\right]^{1/2}$$

$$C(0) = -2i\frac{T\gamma}{m} \left\{ \frac{1}{\omega_+ - \omega_-} \right\} \left\{ \frac{1}{(-2i\omega_+\gamma)} - \frac{1}{(-2i\omega_-\gamma)} \right\}$$

$$= \frac{-2iT\gamma/m}{-2i\gamma}\ \frac{1}{\omega_+ - \omega_-}\ \frac{\omega_- - \omega_+}{\omega_+\omega_-}$$

$$= -\frac{T}{m} \cdot \frac{1}{\omega_+\omega_-}$$

and $\omega_+\omega_- = \left(-\tfrac{1}{2}\gamma + [\ ]^{1/2}\right)\left(-\tfrac{1}{2}\gamma - [\ ]^{1/2}\right)$

$$= -\gamma^2/4 - \omega_0^2 + \gamma^2/4 = -\omega_0^2$$

Finally

$$C(0) = \langle \theta^2 \rangle = \frac{T}{m\omega_0^2}$$

Equipartition implies

$$\langle V \rangle = \langle \tfrac{1}{2} m \omega_0^2 \theta^2 \rangle = \frac{T}{2}$$

the same relation !

2.) a.) $$\frac{d}{dt} \rho_\alpha = +P_{\alpha\beta} \rho_\beta - P_{\beta\alpha} \rho_\alpha$$

$$\frac{d}{dt} \rho_\beta = +P_{\beta\alpha} \rho_\alpha - P_{\alpha\beta} \rho_\beta$$

$$\frac{d}{dt} (\rho_\alpha + \rho_\beta) = 0$$

$$\frac{d}{dt} \rho = M \rho \qquad M = \begin{pmatrix} -P_{\beta\alpha} & P_{\alpha\beta} \\ P_{\beta\alpha} & -P_{\alpha\beta} \end{pmatrix}$$

If $\rho \sim (P_{\alpha\beta}, P_{\beta\alpha})$

then $M\rho = 0$ and the distribution is constant in time. Then

$$\frac{\rho_\alpha}{\rho_\beta} = \frac{P_{\alpha\beta}}{P_{\beta\alpha}}$$ which is <u>detailed</u> <u>balance</u>

In this state

$$\rho_\alpha = \frac{P_{\alpha\beta}}{P_{\alpha\beta}+P_{\beta\alpha}} \qquad \rho_\beta = \frac{P_{\beta\alpha}}{P_{\alpha\beta}+P_{\beta\alpha}}$$

b.) We have already found one right eigenvector of M:

$$\mathbf{v}_0 = (P_{\alpha\beta}, P_{\beta\alpha}) \qquad M v_0 = 0$$

Look for another:

$$M \binom{A}{B} = \lambda_1 \binom{A}{B}$$

$$-P_{\beta\alpha} A + P_{\alpha\beta} B = \lambda_1 A$$
$$+P_{\beta\alpha} A - P_{\alpha\beta} B = \lambda_1 B$$

so $A = -B \qquad v_1 = (1,-1) \qquad \lambda_1 = -(P_{\beta\alpha}+P_{\alpha\beta})$

v_0 is the stationary state

v_1 decays as $\exp[-(P_{\alpha\beta}+P_{\beta\alpha})t]$

For $\rho(0) = (1,0)$

$$= \frac{(P_{\alpha\beta}, P_{\beta\alpha})}{(P_{\alpha\beta}+P_{\beta\alpha})} + \frac{P_{\beta\alpha}\ (1,-1)}{(P_{\alpha\beta}+P_{\beta\alpha})}$$

$$\rho(0) = \frac{1}{P_{\alpha\beta} + P_{\beta\alpha}} v_0 + \left(\frac{P_{\beta\alpha}}{P_{\alpha\beta} + P_{\beta\alpha}}\right) v_1$$

$$\rho(t) = \frac{1}{P_{\alpha\beta} + P_{\beta\alpha}} v_0 + \frac{P_{\beta\alpha}}{P_{\alpha\beta} + P_{\beta\alpha}} (1,-1)\, e^{-(P_{\alpha\beta}+P_{\beta\alpha})t}$$

then

$$\rho_\alpha(t) = \frac{1}{(P_{\alpha\beta} + P_{\beta\alpha})} \left[P_{\alpha\beta} + P_{\beta\alpha}\, e^{-(P_{\alpha\beta}+P_{\beta\alpha})t} \right]$$

$$\rho_\beta(t) = \frac{1}{(P_{\alpha\beta} + P_{\beta\alpha})} P_{\beta\alpha} \left(1 - e^{-(P_{\alpha\beta}+P_{\beta\alpha})t} \right)$$

c.) The system is time translation invariant, so we can take $t' = 0$.

At $t' = 0$, the probability to find the system in α is

$$\bar{P}_\alpha = \frac{P_{\alpha\beta}}{P_{\alpha\beta} + P_{\beta\alpha}}$$

The probability to find the system in β is $\bar{P}_\beta = \dfrac{P_{\beta\alpha}}{P_{\alpha\beta} + P_{\beta\alpha}}$

The average resistance is

$$\bar{R} = \bar{P}_\alpha R_\alpha + \bar{P}_\beta R_\beta$$

At $t=0$, the junction can either be in α or in β
If the junction is in α, then the expected resistance at time t is:

$$\langle R(t;\alpha)\rangle = R_\alpha P_{\alpha\leftarrow\alpha}(t) + R_\beta P_{\beta\leftarrow\alpha}(t)$$

similarly, if the junction is in β at $t'=0$, the expected resistance at time t is

$$\langle R(t;\beta)\rangle = R_\alpha P_{\alpha\leftarrow\beta}(t) + R_\beta P_{\beta\leftarrow\beta}(t)$$

The correlation function of resistance is then

$$\langle R(t)R(0)\rangle = [R_\alpha P_{\alpha\leftarrow\alpha}(t) + R_\beta P_{\beta\leftarrow\alpha}(t)]\cdot \bar{P}_\alpha \cdot R_\alpha$$
$$+ [R_\alpha P_{\alpha\leftarrow\beta}(t) + R_\beta P_{\beta\leftarrow\beta}(t)]\, \bar{P}_\beta\, R_\beta$$

Sethna gives the nice hint that we should now compute

$$\langle (R(t)-R_\beta)(R(0)-R_\beta)\rangle$$

all terms with R_β disappear and we find

$$= (R_\alpha - R_\beta)^2\ \bar{P}_\alpha\ P_{\alpha\leftarrow\alpha}(t)$$

Now we need to find $C(t)$ in terms of this object.

$$C(t) = \langle (R(t) - \bar{R})(R(0) - \bar{R}) \rangle$$

$$= \langle (R(t) - R_\beta)(R(0) - R_\beta) \rangle + \langle (R_\beta - \bar{R})(R(0) - R_\beta) \rangle$$
$$+ \langle (R(t) - R_\beta)(R_\beta - \bar{R}) \rangle + \langle (R_\beta - \bar{R})^2 \rangle$$

$$= \langle (R(t) - R_\beta)(R(0) - R_\beta) \rangle + (R_\beta - \bar{R})(\bar{R} - R_\beta)$$
$$+ (\bar{R} - R_\beta)(R_\beta - \bar{R}) + (R_\beta - \bar{R})^2$$

$$= \langle (R(t) - R_\beta)(R(0) - R_\beta) \rangle - (R_\beta - \bar{R})^2$$

$$= (R_\alpha - R_\beta)^2 \bar{P}_\alpha \; \mathcal{P}_{\alpha\leftarrow\alpha}(t)$$
$$- \left[R_\beta - (\bar{P}_\alpha R_\alpha + \bar{P}_\beta R_\beta)\right]^2 \qquad \bar{P}_\alpha = 1 - \bar{P}_\beta$$

$$= (R_\alpha - R_\beta)^2 \bar{P}_\alpha \; \mathcal{P}_{\alpha\leftarrow\alpha}(t) - \bar{P}_\alpha^2 (R_\alpha - R_\beta)^2$$

so

$$C(t) = (R_\alpha - R_\beta)^2 \; \bar{P}_\alpha \left[\mathcal{P}_{\alpha\leftarrow\alpha}(t) - \bar{P}_\alpha\right]$$

as $t \to \infty$ $\quad \mathcal{P}_{\alpha\leftarrow\alpha}(t) \to \bar{P}_\alpha \quad$ so $\quad C(t) \to 0$

for $t = 0$ $\quad \mathcal{P}_{\alpha\leftarrow\alpha}(t) = 1, \quad C(t) = (R_\alpha - R_\beta)^2 \bar{P}_\alpha \bar{P}_\beta$

Compare this to:

$$\bar{P}_\alpha\ (R_\alpha - \bar{R})^2 + \bar{P}_\beta\ (R_\beta - \bar{R})^2$$

$$= \bar{P}_\alpha\ \left(R_\alpha(1-\bar{P}_\alpha) - P_\beta R_\beta\right)^2 + \bar{P}_\beta\ \left(R_\beta(1-\bar{P}_\beta) - P_\alpha R_\alpha\right)^2$$

$$= \bar{P}_\alpha\ \bar{P}_\beta^2\ (R_\alpha - R_\beta)^2 + \bar{P}_\beta\ \bar{P}_\alpha^2\ (R_\beta - R_\alpha)^2$$

$$= \bar{P}_\alpha \bar{P}_\beta\ \underbrace{(\bar{P}_\beta + \bar{P}_\alpha)}_{=1}\ (R_\alpha - R_\beta)^2$$

$$= \bar{P}_\alpha \bar{P}_\beta\ (R_\alpha - R_\beta)^2 = C(0)\ \checkmark$$

As a final check, substitute the formula for $P_{\alpha\to\alpha}(t)$ into the formula for $C(t)$ and check that the formula is symmetric between α and β:

$$P_{\alpha\to\alpha}(t) = \frac{P_{\alpha\beta} + P_{\beta\alpha}\ e^{-(P_{\alpha\beta}+P_{\beta\alpha})t}}{P_{\alpha\beta} + P_{\beta\alpha}}$$

$$= \bar{P}_\alpha + \bar{P}_\beta\ e^{-\gamma t} \qquad \gamma = P_{\alpha\beta} + P_{\beta\alpha}$$

$$C(t) = (R_\alpha - R_\beta)^2\ \bar{P}_\alpha\ [\ \bar{P}_\alpha + \bar{P}_\beta\ e^{-\gamma t} - \bar{P}_\alpha\ ]$$

$$= (R_\alpha - R_\beta)^2\ \bar{P}_\alpha \bar{P}_\beta\ e^{-(P_{\alpha\beta}+P_{\beta\alpha})t} \qquad !$$

d.) In equilibrium, there should be no net flow of probability between any pair of states. This gives the detailed balance conditions

$$\frac{\bar{P}_\beta}{\bar{P}_\alpha} = \frac{P_{\beta\alpha}}{P_{\alpha\beta}} \qquad \frac{\bar{P}_\beta}{\bar{P}_\gamma} = \frac{P_{\beta\gamma}}{P_{\gamma\beta}} \qquad \frac{\bar{P}_\alpha}{\bar{P}_\gamma} = \frac{P_{\alpha\gamma}}{P_{\gamma\alpha}}$$

where $\bar{P}_\alpha$ $\bar{P}_\beta$ $\bar{P}_\gamma$ are the equilibrium probabilities

then $\bar{P}_\alpha = \dfrac{P_{\alpha\gamma}}{P_{\gamma\alpha}} \bar{P}_\gamma \qquad \bar{P}_\beta = \dfrac{P_{\beta\gamma}}{P_{\gamma\beta}} \bar{P}_\gamma$

$$\frac{\bar{P}_\beta}{\bar{P}_\alpha} = \frac{P_{\beta\gamma}}{P_{\gamma\beta}} \frac{P_{\gamma\alpha}}{P_{\alpha\gamma}} = \frac{P_{\beta\alpha}}{P_{\alpha\beta}}$$

so $\dfrac{P_{\beta\alpha}}{P_{\alpha\beta}} = \dfrac{P_{\beta\gamma} P_{\gamma\alpha}}{P_{\alpha\gamma} P_{\gamma\beta}}$

3.) a.) With periodic b.c.'s, the free energy is

$$F = \int_0^L dx \left\{ \frac{c}{2} (\vec{\nabla} M)^2 + \frac{B}{2} M^2 \right\}$$

Write M as a Fourier series:

$$M(x) = \sum_n \tilde{M}_n e^{i k_n x} \qquad \tilde{M}_n = \frac{1}{L} \int_0^L dx \, M(x) e^{-i k_n x}$$

$$k_n = 2\pi n / L \qquad \tilde{M}_n^* = \tilde{M}_{-n}$$

then

$$F = \int_0^{L} dx \sum_n \tilde{M}_n e^{ik_n x} \sum_{n'} \tilde{M}_{n'} e^{ik_{n'} x}$$

$$\cdot \left(\frac{C}{2}\, ik_n \cdot ik_{n'} + \frac{B}{2} \right)$$

the ~~integral~~ over x sets $k_{n'} = -k_n$ or $n' = -n$

$$F = L \sum_{n=-\infty}^{\infty} \left(\frac{C}{2} k_n^2 + \frac{B}{2} \right) |\tilde{M}_n|^2$$

$\tilde{M}_0$ is real. For $n \neq 0$, $\tilde{M}_n = \mathrm{Re}\, M_n + i\, \mathrm{Im}\, M_n$

$$|M_n|^2 = |\tilde{M}_n|^2 = (\mathrm{Re}\, M_n)^2 + (\mathrm{Im}\, M_n)^2$$

$$F/L = \frac{B}{2} M_0^2 + 2 \sum_{n=1}^{\infty} \frac{1}{2} (Ck_n^2 + B) \left[(\mathrm{Re}\, M_n)^2 + (\mathrm{Im}\, M_n)^2 \right]$$

so F is a sum of terms of the form $\frac{1}{2} A x^2$

so we can apply the equipartition theorem $\langle \frac{1}{2} A x^2 \rangle = \frac{T}{2}$

$$\langle L \frac{B}{2} (M_0^2) \rangle = \frac{T}{2} \Rightarrow M_0^2 = \frac{T}{LB}$$

$$\langle 2L \cdot \tfrac{1}{2} (Ck_n^2 + B)(\mathrm{Re}\, M_n)^2 \rangle = \frac{T}{2}$$

$$\Rightarrow \langle (\mathrm{Re}\, M_n)^2 \rangle = \langle (\mathrm{Im}\, M_n)^2 \rangle = \frac{T}{2L(Ck_n^2 + B)}$$

Now we can compute $\langle \tilde{M}_{-m} \tilde{M}_{n} \rangle$. This is zero unless $m=n$ or $m=-n$. But actually

$$\langle \tilde{M}_m \tilde{M}_m \rangle = \langle (\mathrm{Re}\, M_m)^2 + 2i\, \mathrm{Re} M_m\, \mathrm{Im}\, M_m - (\mathrm{Im}\, M_m)^2 \rangle$$

$$= \langle (\mathrm{Re}\, M_m)^2 \rangle + 0 - \langle (\mathrm{Im}\, M_m)^2 \rangle$$

$$= 0$$

so the correlator is nonzero only for the case

$$\langle \tilde{M}_{\underline{m}} \tilde{M}_m \rangle = \langle (\mathrm{Re}\, M_m - i\, \mathrm{Im}\, M_m)(\mathrm{Re}\, M_m + i\, \mathrm{Im}\, M_m) \rangle$$

$$= \langle (\mathrm{Re}\, M_m)^2 \rangle + \langle (\mathrm{Im}\, M_m)^2 \rangle$$

$$= \frac{T}{L(Ck_n^2 + B)}$$

This agrees with the case $m=0$ and we find in all cases:

$$\langle \tilde{M}_{-m} \tilde{M}_n \rangle = \delta_{mn} \frac{T}{L(Ck_n^2 + B)}$$

b.) $$\langle M(x) M(0) \rangle = \langle \sum_m \tilde{M}_m e^{ik_m x} \sum_n \tilde{M}_n \rangle$$

$$= \sum_m e^{ik_m x} \frac{T}{L(Ck_m^2 + B)}$$

now $k_m = \frac{2\pi m}{L}$ so $\sum_m = \frac{L}{2\pi} \sum_k \Delta k \Rightarrow \frac{L}{2\pi} \int dk$

$$\langle M(x)\,M(0)\rangle = T_c \int \frac{dk}{2\pi}\ \frac{T}{T_c\,(Ck^2+B)}\ e^{ikx}$$

$$= \int \frac{dk}{2\pi}\ \frac{T}{Ck^2+B}\ e^{ikx}$$

To evaluate this, write the expression as a contour integral and (for $x>0$) push the contour upward:

and pick up the pole at $k = +i\left(\frac{B}{C}\right)^{1/2}$

$$\langle M(x)\,M(0)\rangle = 2\pi i\ \frac{1}{2\pi}\ \frac{T}{2C\ i\left(\frac{B}{C}\right)^{1/2}}\ e^{-(B/C)^{1/2}x}$$

$$\langle M(x)\,M(0)\rangle = \frac{T}{2C(B/C)^{1/2}}\ e^{-(B/C)^{1/2}x}$$

The expression must be symmetric under $x \to -x$

$$\langle M(x)\,M(0)\rangle = \frac{T}{2(BC)^{1/2}}\ e^{-(B/C)^{1/2}|x|}$$

c.) $\dfrac{\delta \mathcal{F}}{\delta M}$ is computed by considering

$$\delta \mathcal{F} = \int_0^{L} dx\ \left[C(\nabla M)(\nabla\,\underbrace{\delta M)}_{\text{integrate by parts}} + B\,M\,\delta M\right]$$

$$= \int_0^{L} dx\ \ \delta M\left(-C\nabla^2 M + BM\right)$$

that is,
$$\frac{\delta \mathcal{F}}{\delta M} = - C \nabla^2 M + BM$$

d.) The time-dependent equation for M is then

$$\frac{\partial}{\partial t} M(x,t) = -\eta \, (- C\nabla^2 + B) \, M(x,t)$$

This is close to the diffusion equation. If B were zero, we would have

$$\frac{\partial}{\partial t} M = \eta C \nabla^2 M$$

for which the solution with $M(x, t=0) = \delta(x)$ is

$$M(x,t) = \frac{1}{\sqrt{4\pi\eta C t}} \exp\left[- x^2/4\eta C t\right]$$

With B, the solution is

$$M(x,t) = \frac{1}{\sqrt{4\pi\eta C t}} \, e^{- x^2/4\eta C t} \, e^{-\eta B t}$$

Alternatively, Fourier transform the equation in x to find

$$\frac{\partial}{\partial t} M(k,t) = - \eta \, (C k^2 + B) \, M(k,t)$$

The initial condition is $M(x,0) = \delta(x) = \int \frac{dk}{2\pi} e^{ikx} \cdot 1$

or $M(k,t) = 1$ at $t = 0$

Then $$M(k,t) = e^{-\eta(Ck^2+B)t}$$

$$M(x,t) = \int \frac{dk}{2\pi}\, e^{ikx}\, e^{-\eta(Ck^2+B)t}$$

$$= \int \frac{dk}{2\pi}\, e^{-\eta C\left(k - i\frac{x}{2\sqrt{\eta C t}}\right)^2} e^{-\frac{x^2}{4\eta C t}} e^{-\eta B t}$$

$$= \frac{1}{\sqrt{4\pi\eta C t}}\, e^{-x^2/4\eta C t}\, e^{-\eta B t}$$

in agreement w. p. 16.

e.) Using the Onsager regression hypothesis,

$$\langle M(x,t)\, M(0,0)\rangle = \int dx'\, G(x-x',t)\, \langle M(x')\, M(0)\rangle$$

where G is the result of part (d)

and $\langle M(x)\, M(0)\rangle$ is the result of part (b)

$$\langle M(x,t)\, M(0,0)\rangle = C(x,t)$$

$$= \int_{-\infty}^{\infty} dx'\, \frac{1}{\sqrt{4\pi\eta C t}}\, e^{-(x'-x)^2/4\eta C t}\, e^{-\eta B t}\, \frac{T}{2(BC)^{1/2}}\, e^{-(B/C)^{1/2}|x'|}$$

From here on, assume $x > 0$; the answer must be symmetric under $x \to -x$.

$$C(x,t) = \frac{T}{\sqrt{16\pi\eta BC^2 t}} \, e^{-\eta B t}$$

$$\cdot\left\{ \int_0^\infty dx' \, e^{-(x'-x)^2/4C\eta t} \, e^{-(B/C)^{1/2} x'} + \int_0^\infty dx' \, e^{-(x'+x)^2/4\eta C t} \, e^{-(B/C)^{1/2} x'} \right\}$$

We need

$$\int_0^\infty dx' \, e^{-(x'+x)^2/4\eta C t} \, e^{-(B/C)^{1/2} x'} \qquad x' = y - x$$

$$= \int_x^\infty dy \, e^{-y^2/4\eta C t} \, e^{-(B/C)^{1/2} y} \, e^{+(B/C)^{1/2} x}$$

$$= \int_x^\infty dy \, e^{-\frac{1}{4\eta C t}[y + 2\eta t (BC)^{1/2}]^2} \, e^{\eta B t} \, e^{(B/C)^{1/2} x}$$

$$= \int_{x + 2\eta t (BC)^{1/2}}^\infty dy \, e^{-\frac{y^2}{4\eta C t}} \, e^{\eta B t} \, e^{(B/C)^{1/2} x}$$

$$= (4\eta C t)^{1/2} \left[\int_{z = \frac{x + 2\eta t (BC)^{1/2}}{\sqrt{4\eta C t}}}^\infty dz \, e^{-z^2} \right] e^{\eta B t} \, e^{(B/C)^{1/2} x}$$

Now introduce $\quad \mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-z^2} \, dz$

$$\mathrm{erfc}(z) \to 1 \quad \text{as} \quad z \to 0$$

$$\sim \frac{2}{\sqrt{\pi}} \frac{e^{-z^2}}{2z} \quad \text{as} \quad z \to \infty$$

so

$$\int_0^\infty dx' \; e^{-(x'+x)^2/4\eta Ct} \; e^{-(B/C)^{1/2}x'}$$

$$= \frac{\sqrt{\pi}}{2} (4\eta Ct)^{1/2} e^{(B/C)^{1/2}x} e^{\eta Bt} \operatorname{erfc}\left(\frac{x + 2\eta (BC)^{1/2} t}{\sqrt{4\eta Ct}} \right)$$

similarly

$$\int_0^\infty e^{-(x'-x)^2/4\eta Ct} \; e^{-(B/C)^{1/2}x'}$$

$$= \frac{\sqrt{\pi}}{2} (4\eta Ct)^{1/2} e^{-(B/C)^{1/2}x} e^{\eta Bt} \operatorname{erfc}\left(\frac{2\eta (BC)^{1/2} t - x}{\sqrt{4\eta Ct}} \right)$$

now assemble the pieces. Note that

$$\frac{2\eta (BC)^{1/2} t - x}{\sqrt{4\eta Ct}} = (\eta Bt)^{1/2} - \frac{x}{\sqrt{4\eta Ct}}$$

then!

$$C(x,t) = \frac{T}{4(BC)^{1/2}} \left\{ e^{-(B/C)^{1/2}x} \operatorname{erfc}\left((\eta Bt)^{1/2} - \frac{x}{\sqrt{4\eta Ct}} \right) \right.$$

$$\left. + \; e^{+(B/C)^{1/2}x} \operatorname{erfc}\left((\eta Bt)^{1/2} + \frac{x}{\sqrt{4\eta Ct}} \right) \right\}$$

as $x \to 0$ $\quad C(0,t) = \frac{T}{4(BC)^{1/2}} \cdot 2 \operatorname{erfc}\left((\eta Bt)^{1/2} \right)$

$$\to \frac{T}{2(BC)^{1/2}} \quad \text{as } t \to 0$$

$$\to \frac{T}{(\pi BC)^{1/2}} \frac{1}{2(\eta Bt)^{1/2}} e^{-\eta Bt} \quad \text{as } t \to \infty$$

more generally, as $t \to 0$, $x > 0$

$$\operatorname{erfc}\left((\eta B t)^{1/2} - \frac{x}{\sqrt{4\eta C t}}\right) \to \operatorname{erfc}(-\infty) = 2$$

$$\operatorname{erfc}\left((\eta B t)^{1/2} + \frac{x}{\sqrt{4\eta C t}}\right) \to \operatorname{erfc}(\infty) = 0$$

$$C(x,t) \to \frac{T}{2(BC)^{1/2}} e^{-(B/C)^{1/2} x} \quad \checkmark$$

as $t \to \infty$ $x > 0$

$$\operatorname{erfc}\left((\eta B t)^{1/2} \pm \frac{x}{\sqrt{4\eta C t}}\right) \sim \frac{2}{\sqrt{\pi}} \frac{1}{2(\eta B t)^{1/2}} e^{-\left((\eta B t)^{1/2} \pm \frac{x}{(4\eta C t)^{1/2}}\right)^2}$$

$$\sim \frac{1}{\sqrt{\pi}} \frac{1}{(\eta B t)^{1/2}} e^{-\eta B t} e^{\mp (B/C)^{1/2} x} e^{-x^2/4\eta C t}$$

so

$$C(x,t) \underset{t\to\infty}{\longrightarrow} \frac{T}{2(BC)^{1/2}} \frac{1}{(\pi \eta B t)^{1/2}} e^{-\eta B t} e^{-x^2/4\eta C t}$$

f.) $$X(x,t) = -\beta \frac{\partial}{\partial t} C(x,t)$$

We can differentiate the formula on p. 19 using

$$\frac{d}{dz} \operatorname{erfc}(z) = -\frac{2}{\sqrt{\pi}} e^{-z^2}$$

then

$$\chi(x,t) = -\frac{1}{4(BC)^{1/2}} \Big\{ e^{-(B/C)^{1/2}x} \left(+\frac{1}{2t}\right)\left((\eta Bt)^{1/2} + \frac{x}{\sqrt{4\eta Ct}}\right)$$

$$-\frac{2}{\sqrt{\pi}} \exp\left[-\left[(\eta Bt)^{1/2} - \frac{x}{(4\eta Ct)^{1/2}}\right]^2\right]$$

$$+ \quad (x)\to(-x) \Big\}$$

the exponential is

$$\exp\Big[-\eta Bt + \underbrace{\left(\tfrac{B}{C}\right)^{1/2}x}_{} - \frac{x^2}{4\eta Ct}\Big]$$

this term cancels the $e^{-(B/C)^{1/2}x}$ in the prefactor

then

$$\chi(x,t) = \frac{1}{2(\pi BC)^{1/2}} \Big\{ \frac{1}{2t}\left((\eta Bt)^{1/2} + \frac{x}{\sqrt{4\eta Ct}}\right) e^{-\eta Bt} e^{-x^2/4\eta Ct}$$

$$+ \quad x\to -x$$

$$= \frac{1}{2(\pi BC)^{1/2}} \Big\{ \frac{1}{2t}(\eta Bt)^{1/2} \cdot 2 \cdot e^{-\eta Bt} e^{-x^2/4\eta Ct} \Big\}$$

so

$$\chi(x,t) = \left(\frac{\eta}{4\pi Ct}\right)^{1/2} e^{-\eta Bt} e^{-x^2/4\eta Ct}$$

4.) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + d^2 \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2}$$

look for solutions of the form

$$u(x,t) = \mathrm{Re}\{A\, e^{ikx}\, e^{-i\Omega t}\}$$

$$-\Omega^2 A = -c^2 k^2 A + d^2 (-i\Omega)(-k^2) A$$

$$\Omega^2 + i d^2 \Omega k^2 - c^2 k^2 = 0$$

$$(\Omega + \tfrac{i}{2} d^2 k^2)^2 + \frac{d^4 k^4}{4} = c^2 k^2$$

$$\Omega = -\frac{i}{2} d^2 k^2 \pm \left[c^2 k^2 - \frac{d^4 k^4}{4}\right]^{1/2}$$

For k sufficiently small, $\lambda = \frac{2\pi}{k}$ large, the damping is a small perturbation

$$\Omega = \pm \omega_k - i\,\Gamma_k$$

$$\omega_k = ck\left(1 - \frac{d^4 k^2}{4c^2}\right)^{1/2}$$

$$\Gamma_k = \frac{d^2 k^2}{2}$$

Then the energy of the wave decays as

$$e^{-d^2k^2t}$$

The damping time is $T_d = \frac{1}{d^2k^2}$

The oscillation period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{ck}\left(1 - \frac{d^4k^2}{4c^2}\right)^{-1/2}$

so

$$Q = \frac{\omega_k}{2I_k} = \frac{ck\left(1 - \frac{d^4k^2}{4c^2}\right)^{1/2}}{d^2k^2}$$

or

$$Q = \frac{c}{d^2k}\left(1 - \frac{d^4k^2}{4c^2}\right)^{1/2} \sim \frac{1}{k} \quad \text{as } k \to 0$$

b.) Re ω_k vanishes at

$$c^2k^2 = \frac{d^4k^4}{4} \quad \text{or} \quad k = \frac{2c}{d^2}$$

For higher k, the system is overdamped.

c.) Now study the forced equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} - d^2\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} = \frac{f(x,t)}{\rho}$$

In Fourier space,

$$\left[-\omega^2 + c^2k^2 + d^2k^2(-i\omega)\right] u(k,\omega) = \frac{1}{\rho} f(k,\omega)$$

so

$$u(k,\omega) = \frac{-1/\rho}{\omega^2 - c^2k^2 + i\, d^2\omega k^2} f(k,\omega)$$

We can identify

$$\chi(k,\omega) = \frac{(-1/\rho)}{\omega^2 - c^2k^2 + i d^2\omega k^2}$$

The poles of this expression are the solutions of

$$\omega^2 + i d^2\omega k^2 - c^2k^2 = 0$$

which we derived on p. 22. They are located at

$$\omega = \pm\,\omega_k - i\,\Gamma_k$$

where $\omega_k \to ck$ as $d \to 0$ and $\Gamma_k = \frac{d^2k^2}{2}$

d.) In the underdamped case, the poles are obviously in the lower half plane. In the overdamped case $k > \frac{2c}{d^2}$

$$\omega = -\tfrac{1}{2}d^2k^2 \pm i\left[\left(\frac{d^2k^2}{2}\right)^2 - c^2k^2\right]^{\frac{1}{2}}$$

and even for the + solution $\mathrm{Im}\,\omega < 0$.

e.) $$\chi(k,\omega) = \frac{(-1/\rho)}{(\omega^2 - c^2k^2 + i\omega d^2k^2)} \frac{(\omega^2 - c^2k^2 - i\omega d^2k^2)}{(\omega^2 - c^2k^2 - i\omega d^2k^2)}$$

$$= \frac{(+1/\rho)\left[(c^2k^2 - \omega^2) + i\,\omega d^2k^2\right]}{(\omega^2 - c^2k^2)^2 + (\omega d^2k^2)^2}$$

$$\chi'' = \frac{\omega d^2k^2}{\rho\left[(\omega^2 - c^2k^2)^2 + (\omega d^2k^2)^2\right]}$$

$$C(k,\omega) = \frac{2}{\beta\omega}\chi''(k,\omega) = \frac{2T\ d^2k^2}{\rho\left[(\omega^2 - c^2k^2)^2 + (\omega d^2k^2)^2\right]}$$

for the material in question

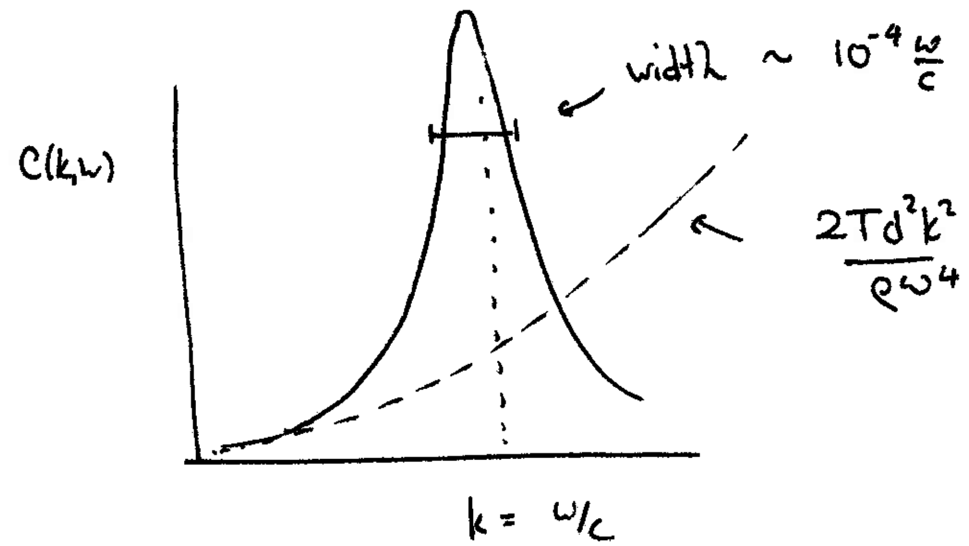
$c = 10^3\,m/sec$ so $\omega = 10^8/sec \rightarrow k = 10^5/m$

$d = 10^{-3}\,m/sec^{1/2}$ so

$$\Gamma_k = \frac{d^2k^2}{2} \sim 10^4/sec \sim 10^{-4}\,\omega.$$

The shape of $C(k,\omega)$ at fixed ω, varying k is

then $\longrightarrow$



Probably it is difficult to measure the width of the resonance, but d can be obtained from the normalization of $C(k,\omega)$ off the resonance.